Derivation of a two-fluid flow model from a kinetic formulation

An attempt to derive an immiscible two-fluid model

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Context and motivation

Is it possible to derive a **macroscopic model of immiscible two-fluid flows** using a **microscopic kinetic description**?

Take advantage of the **multi-species Boltzmann model** [4]

- Mixture of gases (miscible phases by nature)
- Fluid equations accounting for
  - Relaxation source terms towards thermodynamical equilibrium
  - Velocity disequilibrium

Difficulties and aims

✗ How to model the immiscible behavior?

✗ Well-posedness properties

Modelling: microscopic description of the two fluids

System of two types of fluids $i \in \{-1,+1\}$, of mass m_i, with strong repulsive force

- **Density of matter** $f_i(t,x,v)$ describes the species $i \in \{-1,+1\}$ where $x \in \mathbb{R}^d_{2\pi-\mathrm{per}}$ or $x \in \Omega$ smooth bounded and connected, $v \in \mathbb{R}^d$, $t \in \mathbb{R}^+$
- **Mixture and fluid quantities of matter**
  - Fluid densities $\rho_i(t,x) = \int_v f_i(t,x,v)\mathrm{d}v$ and momentum $\rho_i u_i(t,x) = \int_v v f_i(t,x,v)\mathrm{d}v$
  - Mixture density $\rho(t,x) = \sum_{i=\pm1} \rho_i(t,x)$ and momentum $\rho u(t,x) = \sum_{i=\pm1} \rho_i u_i(t,x)$
- **Mean field potential** V

$$V_i(t,x) = K(x) \star \rho_i(t,x)$$

  - K smooth nonnegative even function with compact support (ball of radius r)

⇝ **Repulsion** from particles $i = \pm1$ on $\bar{i} = -i$

Kinetic equation

(KE)
$$\partial_t f_i + v\cdot\nabla_x f_i - \frac{1}{\eta}\frac{1}{m_i}\nabla_x V_{\bar{i}}\cdot\nabla_v f_i = \frac{1}{\varepsilon}\big(Q_{ii}(f_i,f_i) + Q_{i\bar{i}}(f_i,f_{\bar{i}})\big)$$

- Fluid i strongly repelled by fluid $\bar{i}$ *via* the force term $-\frac{1}{\eta}\nabla_x V_{\bar{i}}$, scaling parameter $\eta \ll 1$
- Attraction within each fluid is neglected (no additional difficulties)
- Knudsen number $\varepsilon \ll 1$

- **Collision operator**
  - Bilinear collisional operators Q_{ii} and $Q_{i\bar{i}}$ of Boltzmann type with simple collision kernel

$$Q_{ij}(f_i,f_j)(v) = \int_{\mathbb{R}^3\times\mathbb{S}^2} B_{ij}(|v-v^*|,\cos\theta)(f_i' f_j'^* - f_i f_j^*)\mathrm{d}v_*\mathrm{d}\sigma, \qquad i,j\in\{-1,+1\}$$

  - Cross section $B_{ij}(|v-v^*|,\cos\theta) = b_{ij}(\cos\theta)$ with an even $b_{ij} \in L^1$ [3]
  - $(f_i' f_j'^* - f_i f_j^*)$ ensures conservation of moment and energy (weighted by fluid masses)
  - **Conservation within the fluid** i: mass, momentum and energy

$$\int_v Q_{i,i}(f_i,f_i)\begin{pmatrix}1\\ m_i v\\ m_i\frac{v^2}{2}\end{pmatrix}\mathrm{d}v = 0 \qquad i\in\{-1,+1\}$$

  - **Interspecies momentum and energy conservation**

$$\forall i\in\{-1,+1\},\quad \int_v Q_{i,\bar{i}}(f_i,f_{\bar{i}})\mathrm{d}v = 0 \quad \text{and} \quad \sum_{i=\pm1}\int_v Q_{i,\bar{i}}(f_i,f_{\bar{i}})\begin{pmatrix}m_i v\\ m_i\frac{v^2}{2}\end{pmatrix}\mathrm{d}v = 0$$

Well-posedness properties

Energy conservation

Consider a sufficiently smooth *a priori* solution of (KE), the energy

$$E(\eta) = \sum_{i=\pm1}\int_x\int_v \frac{1}{2}m_i|v|^2 f_i(t,x,v)\mathrm{d}v\mathrm{d}x + \frac{1}{\eta}\sum_{i=\pm1}\frac{1}{2}\int_x V_{\bar{i}}(t,x)\rho_i(t,x)\mathrm{d}x$$

is conserved in time

- Idea of the proof
  - Consider the first momentum of (KE) and conservation properties of the collisional operator
  - Take advantage of the parity of K to handle the force term

✓ $E(\eta)$ is a sum of positive terms

✗ **Further main assumption**: $E(\eta)$ is bounded uniformly in η

Entropy dissipation

- Entropy of the system

$$H(f) = \sum_{i=\pm1}\int_x\int_v f_i\ln(f_i)\mathrm{d}x\mathrm{d}v$$

- Entropy dissipation

$$D_{ij}(f_i,f_j)(v) = \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{S}^3} b_{ij}(\cos\theta)(f_i' f_j'^* - f_i f_j^*)\ln\left(\frac{f_i' f_j'^*}{f_i f_j^*}\right)\mathrm{d}v_*\mathrm{d}\sigma \geq 0, \quad i,j\in\{-1,+1\}$$

It holds

$$\frac{d}{dt}\sum_{i=\pm1}\int_x\int_v f_i\ln(f_i)\mathrm{d}v\mathrm{d}x = -\frac{1}{\varepsilon}\sum_{i=\pm1}\int_x\int_v \big(D_{ii}(f_i,f_i) + D_{i\bar{i}}(f_i,f_{\bar{i}})\big)\mathrm{d}v\mathrm{d}x \leq 0$$

Existence results

- For given ε and η, there exist (ultra) weak solutions of (KE) [5]
- Under the assumption of uniform boundedness of $E(\eta)$, energy conservation and entropy dissipation ensure compactness for the family of solutions of (KE) indexed by η and ε. In particular, when ε and/or $\eta \to 0$, hydrodynamical limit can be studied: under moment convergence assumptions, there exist very weak solutions of (KE) [2, 1]

Macroscopic equations for a given η

Taking successive moments in v of the kinetic equation (KE), one exhibits the fluid equations of motion

- Fluid pressure p_i and heat flux q_i : $p_i^{(\ell k)} = \int_v (v^{(k)} - u_i^{(k)})(v^{(\ell)} - u_i^{(\ell)})f_i(t,x,v)\mathrm{d}v, \quad k,\ell \leq d$
- Fluid internal energy e_i: $\frac{1}{2}\int_v m_i\frac{v^2}{2}f_i(t,x,v)\mathrm{d}v = \frac{1}{2}m_i\frac{u_i^2}{2} + m_i\rho_i e_i$

Fluid equations

- Mass equations

$$\partial_t\rho_i + \mathrm{div}(\rho_i u_i) = 0$$

- Momentum equations

$$\partial_t(\rho_i u_i^{(\ell)}) + \sum_{k\leq d}\partial_{x_k}\big(\rho_i u_i^{(k)}u_i^{(\ell)} + p_i^{(k\ell)}\big) + \frac{1}{\eta m_i}\rho_i\partial_{x_k}V_{\bar{i}} = \frac{m_{\bar{i}}}{m_i + m_{\bar{i}}}2\pi\|b_{i\bar{i}}\|_{L^1}\rho_i\rho_{\bar{i}}(u_{\bar{i}}^{(\ell)} - u_i^{(\ell)})$$

- Energy equations

$$\partial_t\Big(\frac{1}{2}\rho_i u_i^2 + \rho_i e_i\Big) + \sum_{k\leq 3}\partial_{x_k}\Big(u_i^{(k)}\big(\frac{1}{2}\rho_i u_i^2 + \rho_i e_i\big) + \sum_{\ell\leq 3}u_i^{(\ell)}p_i^{(k\ell)} + q_i^{(k)}\Big) + \frac{1}{\eta m_i}\rho_i u_i^{(k)}\partial_{x_k}V_{\bar{i}}$$
$$= 4\pi\frac{\|b_{i\bar{i}}\|_{L^1}m_{\bar{i}}}{(m_i + m_{\bar{i}})^2}\rho_i\rho_{\bar{i}}\big((m_{\bar{i}}u_{\bar{i}} + m_i u_i)(u_{\bar{i}} - u_i) + 2(m_{\bar{i}}e_{\bar{i}} - m_i e_i)\big)$$

Perspectives

- Hydrodynamical limit (compressible Euler), Maxwellian ansatz (fixed η and $\varepsilon \to 0$): $f_i(t,x,v) = \frac{m_i^{3/2}\rho_i(t,x)}{(2\pi T_i)^{3/2}}e^{-m_i|v-u_i|^2/2T_i} \rightsquigarrow p_i^{(k\ell)} = 0$ and $m_i p_i^{(\ell\ell)} = 2m_i\rho_i e_i/3 = \rho_i T_i$
- At the limit $\eta \to 0$, we have $\rho_i(K \star \rho_{\bar{i}}) = 0$ which means separation of fluids with a void zone of size $r \ll 1$. Study of the logistic equation (of each fluid, r is the size of support of K)
- Formal/rigorous limits with respect to the parameters ϵ, η and possibly r (depending of each others?)
- Long time behavior, other scalings (Navier-Stokes, incompressible Euler), boundary conditions...
- Making the link with Baer-Nunziato models, homogenization, patterns...

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