Excursion sets over high levels of non-Gaussian infinitely divisible random fields: extreme values, algebra, and geometry

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Let $\big(X(\mathbf{t}), \mathbf{t} \in T\big)$ be a random field indexed by some parameter space T.

In many applications, including statistical hypothesis testing, one is interested in, whether or not, the random field crosses a level u, often a high level $u > 0$.

Specifically: suppose we use a test statistic $\sup_{\mathbf{t} \in T} X(\mathbf{t})$.

Let $u > 0$ be so large that, under the null hypothesis on the probability law of the random field $\big(X(\mathbf{t}), \mathbf{t} \in T\big)$, the level crossing probability $P\big(\sup_{\mathbf{t} \in T} X(\mathbf{t}) > u\big)$ is small.

Decision rule: reject the null hypothesis if $\sup_{\mathbf{t} \in T} X(\mathbf{t}) > u$.

The excursion set

The *excursion set* of the random field $\big(X(\mathbf{t}),\, \mathbf{t} \in T\big)$ over the level u is the random set

$$A_u = \big\{\mathbf{t} \in T : X(\mathbf{t}) > u\big\}\,.$$

The statisical decision rule $\sup_{\mathbf{t}\in T} X(\mathbf{t}) > u$ can be reformulated:

reject the null hypothesis if A_u is non-empty.

More powerful statistical tests can be potentially constructed if one uses more delicate algebraic and/or geometric properties of the excursion set A_u other than it being empty or not.

One of the very useful geometric characteristics of a set is its Euler characteristic.

For a "nice" set A the Euler characteristic $\chi(A)$ depends only the *homotopy type* of the set A:

- for a "nice" one-dimensional set A, $\chi(A) =$ number of connected components (intervals) in A;
- for a "nice" two-dimensonal set A, $\chi(A) =$ number of connected components in A minus the number of "holes";
- for a three-dimensonal set A, $\chi(A) =$ number of connected components in A minus the number of "handles" plus the number of "holes', etc.

Smooth Gaussian random fields

Let T be a "nice" manifold, and let $\big(X(\mathbf{t}), \mathbf{t} \in T\big)$ be a smooth zero mean constant variance σ^2 Gaussian random field on T.

Smoothness assumption includes existence of two continuous partial derivatives plus other verifiable technical assumptions.

Consider the excursion set

$$A_u = \left\{ \mathbf{t} \in T : X(\mathbf{t}) \geq u \right\}.$$

Then an explicit non-asymptotic expression for the expected Euler characteristic of the excursion set exists.

For $u > 0$,

$$E\big(\chi(A_u)\big) = \sum_{j=0}^{N} \mathcal{L}_j(T)\rho_j(u/\sigma)$$

(Adler and Taylor (2007)), where N is the dimension of the manifold, and for $j \geq 1$,

$$\rho_j(u) = (2\pi)^{-(j+1)/2} H_{j-1}(u) e^{-u^2/2}$$

with H_k being the kth Hermite polynomial.

Further,

$$\rho_0(u) = \Psi(u) = \int_u^\infty (2\pi)^{-1/2} e^{-x^2/2}\, dx$$

is the standard normal tail.

The coefficients $\mathcal{L}_j(T)$, $j = 0, \ldots, N$ are the so-called *Lipschitz-Killing curvatures* of T. They are independent of u.

In general, the coefficients $\big(\mathcal{L}_j(T)\big)$ depend on the covariance function of the random field. However: if the random field is isotropic, the coefficients depend only on the manifold T and the second spectral moment of the field.

Much effort is being put into computation of the Lipschitz-Killing curvatures for various manifolds.

The expected Euler characteristic of the excursion set of a smooth Gaussian randon field provides an excellent approximation to the level crossing probability $P\big(\sup_{\mathbf{t}\in T} X(\mathbf{t}) > u\big)$.

Let T be a "nice" manifold, and let $\big(X(\mathbf{t}),\, \mathbf{t}\in T\big)$ be a smooth zero mean constant variance σ^2 Gaussian random field on T. Then there is a number $a>0$ such that for $u>0$ large enough,

$$\left| P\big(\sup_{\mathbf{t}\in T} X(\mathbf{t}) > u\big) - E\big(\chi(A_u)\big)\right| \leq e^{-(1+a)u^2/2\sigma^2}$$

(Taylor, Takemura, Adler (2005)).

That is, the approximation is superexponentially good.

The reason the approximation of the level crossing probability by the expected Euler characteristic of the excursion set is so good is that, for smooth Gaussian random fields, exceedance of a high level u occurs *locally*:

- the maximum of the field, $\sup_{\mathbf{t}\in T} X(\mathbf{t})$, is achieved at a unique point, with probability 1;
- if $\sup_{\mathbf{t}\in T} X(\mathbf{t}) > u$, the excursion set A_u is very likely to be a "nice ball-like" neighbourhood of the point where the maximum is achieved;
- away from a neighborhood of that point the values of the process are very likely to be below u.

Therefore, the excursion set A_u over a high level u, if non-empty, is very likely to have a single connected component, without any holes or other algebraic/geometric features, and its Euler characteristic is very likely to be equal to 1. In particular,

$$E\big(\chi(A_u)\big) = E\Big(\chi(A_u)\mathbf{1}\big(A_u \neq \emptyset\big)\Big)$$

$$\approx P\big(A_u \neq \emptyset\big) = P\big(\sup_{\mathbf{t}\in T} X(\mathbf{t}) > u\big)\,,$$

and the approximation $\approx$ is very good.

This points to certain drawbacks of smooth Gaussian random fields as models in the situations where high level crossings are important.

- The excursion set is nearly always of the same shape, algebraically the same as a ball.
- Very precise information is available about the expected Euler characteristic of the excurson set, but not other features of the Euler characteristic (e.g. its variance).
- The level crossing probabilities are very small for high levels u.

Non-Gaussian infinitely divisible random fields

In order to obtain a tractable class of models with a richer structure of high level excursion sets, we will look at non-Gaussian infinitely divisible random fields. For simplicity of notation, in this talk we will look at random fields on a unit cube, $\big(X(\mathbf{t}), \mathbf{t} \in [-1,1]^d\big)$ and study their excursion sets over a high level.

For technical reasons, we would like to have a random field defined on a neigborhood of the unit cube, and have "nice" properties there.

Let G be an open set in $\mathbb{R}^d$ such that $[-1,1]^d \subset G$. We consider an infinitely divisible random field of the form

$$X(\mathbf{t}) = \int_S f(s;\mathbf{t})\, M(ds),\ \mathbf{t} \in G,$$

where

- M is an infinitely divisible random measure on a measurable space $(S, \mathcal{S})$;
- $f(\cdot;\mathbf{t})$, $\mathbf{t} \in G$ is a family of nonrandom measurable functions on S satisfying certain integrability assumptions.

An infinitely divisible random measure is characterized by its Lévy measure. This is a σ-finite measure on $S \times \big(\mathbb{R} \setminus \{0\}\big)$ of the form

$$F(A) = \int_S \rho\big(s; A_s\big)\, m(ds)$$

for a measurable $A \subset S \times \big(\mathbb{R} \setminus \{0\}\big)$, where

- $A_s = \big\{x \in \mathbb{R} \setminus \{0\} : (s, x) \in A\big\}$ is the s-section of the set A;
- m is a σ-finite measure on $(S, \mathcal{S})$, the so-called control measure of the random measure M;
- $\big(\rho(s; \cdot)\big)$ is a measurable family of Lévy measures on $\mathbb{R}$, the so-called local Lévy measures.

We will assume that the local Lévy measures of the infinitely divisible random measure M possess the following regular variation property:

there exists a function $H : (0,\infty) \to (0,\infty)$ *that is regularly varying at infinity with exponent* $-\alpha$, $\alpha > 0$, *and nonnegative measurable functions* w_+ *and* w_- *on* S *such that*

$$\lim_{u\to\infty} \frac{\rho\big(s;(u,\infty)\big)}{H(u)} = w_+(s), \quad \lim_{u\to\infty} \frac{\rho\big(s;(-\infty,-u)\big)}{H(u)} = w_-(s).$$

In combination with certain integrability assumptions on the kernel $f(\cdot;\mathbf{t})$, $\mathbf{t} \in G$, this means that for every $\mathbf{t} \in G$,

$$\lim_{u\to\infty} \frac{P\big(X(\mathbf{t}) > u\big)}{H(u)} = c_+(\mathbf{t}), \quad \lim_{u\to\infty} \frac{P\big(X(\mathbf{t}) < -u\big)}{H(u)} = c_-(\mathbf{t})$$

for some $c_+, c_- \geq 0$ and, moreover,

$$\lim_{u\to\infty} \frac{P\big(\sup_{\mathbf{t}\in[-1,1]^d} X(\mathbf{t}) > u\big)}{H(u)} = c_{\max},$$

with

$$c_{\max} = \int_S \Big[w_+(s) \sup_{\mathbf{t}\in[-1,1]^d} f(s,\mathbf{t})_+^\alpha + w_-(s) \sup_{\mathbf{t}\in[-1,1]^d} f(s,\mathbf{t})_-^\alpha \Big]\, m(ds).$$

Critical points

For a smooth non-Gaussian infinitely divisible random field we will study the Euler characterstic of its excursion A_u by first studying the critical points of the random field over the level u.

- A critical point of a smooth function f on $\mathbb{R}^d$ is a point where the gradient of f vanishes.
- Critical points can be of different types, depending on the Hessian matrix at the critical point.
- Numbers of critical points of a function above a level are related to the Euler characteristic of the excursion set above the level through the *Morse theory*.

We need to take into account that the notion of a type of a critical point differs from face to face of the cube $[-1,1]^d$.

Let $\mathcal{J}_k$ be the collection of faces of dimension $k = 0, 1, \ldots, d$ of the cube $[-1,1]^d$. For each face $J \in \mathcal{J}_k$ there is a corresponding set $\sigma(J) \subseteq \{1, 2, \ldots, d\}$ of cardinality k and a sequence $\epsilon(J) \in \{-1, 1\}^{\{1,2,\ldots,d\}\setminus\sigma(J)}$ such that

$$J = \Big\{ \mathbf{t} = (t_1, \ldots, t_d) \in [-1,1]^d : \ t_j = \epsilon_j \ \text{ for } j \notin \sigma(J)$$

$$\text{and } \ 0 < t_j < 1 \ \text{ for } j \in \sigma(J) \Big\} .$$

Let g be a "nice" C^2 function on an open set G containing the cube $[-1,1]^d$. For $J \in \mathcal{J}_k$ and $i = 0, 1, \ldots, k$ let $\mathcal{C}_g(J; i)$ be the set of points $\mathbf{t} \in J$ satisfying the following conditions.

$$\frac{\partial g}{\partial t_j} = 0 \quad \text{for each } j \in \sigma(J)\,,$$

$$\epsilon_j \frac{\partial g}{\partial t_j} > 0 \quad \text{for each } j \notin \sigma(J)\,,$$

$$\text{the matrix } \quad \left(\frac{\partial^2 g}{\partial t_m \partial t_n}\right)_{m,n \in \sigma(J)} \quad \text{is non-degenerate}$$

and has i negative eigenvalues.

For $u \in \mathbb{R}$, $k = 0, 1, \ldots, d$, $J \in \mathcal{J}_k$ and $i = 0, 1, \ldots, k$ let

$$\mu_g(J; i : u) = \mathrm{Card}\big(\mathcal{C}_g(J; i) \cap \{\mathbf{t} : \, g(\mathbf{t}) > u\}\big)$$

be the numbers of the critical points of different types and on different faces of the cube, of function g over the level u.

Let

$$A_u(g) = \big\{\mathbf{t} \in [-1, 1]^d : \, g(\mathbf{t}) \geq u\big\}$$

be the excursion set of the function g over the level u. Then

$$\chi\big(A_u(g)\big) = \sum_{k=0}^{d} \sum_{J \in \mathcal{J}_k} \sum_{i=0}^{k} (-1)^i \mu_g(J; i : u) \, .$$

The goal: obtain the limiting conditional joint distribution of the numbers of the critical points of different types and on different faces of the cube of a non-Gaussian infinitely divisible random field over a high level u given that the random field does exceed that level.

This will allow us to compute, for example, the limiting conditional *distribution* of the Euler characteristic of the excursion set of the level u, given that the level is exceeded.

Knowing **the full limiting conditional distribution** we can compute the conditional mean of the Euler characteristic, conditional variance, etc.

The approach: the assumptions on the random field, particularly the assumption of regular variation, imply that, at high levels, the excursions of the random field

$$X(\mathbf{t}) = \int_S f(s;\mathbf{t})\, M(ds),\ \mathbf{t} \in [-1,1]^d$$

are similar to the excursions of a much simpler random field,

$$Z(\mathbf{t}) = Xf(W,\mathbf{t}),\ \mathbf{t} \in [-1,1]^d\,,$$

where $(W,X) \in S \times (\mathbb{R} \setminus \{0\})$ is a random pair whose law is the finite restriction of the Lévy measure F to the set

$$\left\{(s,x) \in S \times (\mathbb{R} \setminus \{0\}) :\ \sup_{\mathbf{t}\in G} |xf(s;\mathbf{t})| > 1\right\},$$

normalized to be a probability measure on that set.

Main theorem

- For $k = 0, 1, \ldots, d$, a face $J \in \mathcal{J}_k$ and $i = 0, 1, \ldots, k$ let $c_i(J; s) = \mu_{f(s;\cdot)}(J; i)$ be the number of the critical points of the s-section of f of the appropriate type.
- Let $\left(\mathbf{t}_l(J; i; s),\ l = 1, \ldots, c_i(J; s)\right)$ be an enumeration of these critical points.
- Let $\left(f_{[m]}^{(J;i:+)}(s),\ m = 1, 2, \ldots\right)$ be the mth largest of the positive parts $\left(f(s; \mathbf{t}_l(J; i; s))\right)_+,\ l = 1, \ldots, c_i(J; s)$;
- let $\left(f_{[m]}^{(J;i:-)}(s),\ m = 1, 2, \ldots\right)$ be the mth largest of the negative parts.

Theorem Assume that an infinitely divisible random field $\big(X(\mathbf{t}),\, \mathbf{t} \in [-1,1]^d\big)$ is smooth, satisfies the assumption of regular variation and additional technical assumtions. Then for any numbers
$n(J;i) = 0,1,2,\dots,\ J \in \mathcal{J}_k,\ k = 0,1,\dots,d,\ i = 0,1,\dots,k$

$$\lim_{u\to\infty} P\Big(\mu_X(J;i:u) \geq n(J;i),\ J \in \mathcal{J}_k,$$

$$k = 0,1,\dots,d,\ i = 0,1,\dots,k \Big| \sup_{\mathbf{t}\in[-1,1]^d} X(\mathbf{t}) > u\Big)$$

$$= \frac{\int_S \Big[w_+(s)\Big(\min_{J,i} f^{(J;i:+)}_{[n(J;i)]}(s)\Big)^\alpha + w_-(s)\Big(\min_{J,i} f^{(J;i:-)}_{[n(J;i)]}(s)\Big)^\alpha\Big]\, m(ds)}{\int_S \Big[w_+(s) \sup_{\mathbf{t}\in[-1,1]^d} f(s,\mathbf{t})^\alpha_+ + w_-(s) \sup_{\mathbf{t}\in[-1,1]^d} f(s,\mathbf{t})^\alpha_-\Big]\, m(ds)}$$

Moving average random fields

A moving average random field is the random field

$$X(\mathbf{t}) = \int_{\mathbb{R}^d} g(\mathbf{s}+\mathbf{t})\, M(d\mathbf{s}), \;\; t \in G,$$

where the control measure of the infinitely divisible random measure M is the d-dimensional Lebesgue measure, and the local Lévy measure $\rho(\mathbf{s},\cdot) = \rho(\cdot)$ is independent of $\mathbf{s} \in \mathbb{R}^d$.

The measure $\rho(\cdot)$ is assumed to be regularly varying.

Choosing the kernel g of different shapes, one can obtain very different algebraic/geometrical properties of the high level excursion sets.

Example 1. Let $g(\mathbf{t}) = \exp\{-\|\mathbf{t}\|^2/2\}$, $\mathbf{t} \in \mathbb{R}^d$. Because of the rotational invariance of this kernel and its radial monotonicity, the high level excursion set A_u, if not empty, looks, algebraically, "like a ball" and, hence, Euler characteristic equal to 1, as in the case of smooth Gassian random fields.

Example 2. Let us take $d = 1$ and

$$g(t) = \big(1 + \cos \gamma t\big) e^{-t^2/2},\ t \in \mathbb{R}\,.$$

In this case the high level excursion set will have a random number of "holes", and the Euler characteristic a non-degenerate conditional distribution.