Topics on Stable Random Fields
Decomposability and Ergodicity

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Motivation: decomposability and ergodicity

Theorem (Rosiński 1995)
Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stationary symmetric α-stable (SαS) process, then

$$\{X_t\}_{t\in\mathbb{Z}} \stackrel{\mathrm{d}}{=} \{X_t^C + X_t^D\}_{t\in\mathbb{Z}},$$

where $\{X_t^C\}_{t\in\mathbb{Z}}$, $\{X_t^D\}_{t\in\mathbb{Z}}$ are independent stationary SαS.

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Revealing connection to conservative/dissipative flows.

Decomposition or classification.

Decomposability of distributions

Given a random variable X, when can we write

$$X \stackrel{\mathrm{d}}{=} Y + Z,$$

with independent random variables Y and Z?

Well known cases: Gaussian, Poisson, and many others.
Self-decomposability, infinite divisibility, sum-stability, etc.

Sum-stable distributions

- X is infinitely divisible, if $\forall n$, one has

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for certain i.i.d. Y_i.

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  where X, X_i's are i.i.d.
- Symmetric α-stable (SαS) laws

$$X \stackrel{\mathrm{d}}{=} n^{-1/\alpha}(X_1 + \cdots + X_n),$$

  with

$$\mathbb{E}\mathrm{e}^{i\theta X} = \mathrm{e}^{-\sigma^\alpha |\theta|^\alpha}, \quad \theta \in \mathbb{R}, \alpha \in (0,2].$$

SαS random fields

A random field $\{X_t\}_{t\in T}$ is SαS, if

$$\sum_{i=1}^{n} a_i X_{t_i} \sim \mathrm{S}\alpha\mathrm{S}$$

for all $a_i \in \mathbb{R}, t_i \in T$.

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- $\alpha = 2 \Rightarrow$ zero-mean Gaussian random field.
- We focus on $\alpha \in (0, 2)$.
- Charaterization via characteristic function of finite-dimensional distribution (f.d.d.).

Representation of SαS random fields

Given an SαS random field $\{X_t\}_{t\in T}$, $\exists$ certain $\{f_t\}_{t\in T} \subset L^\alpha(S,\mu)$,

$$\mathbb{E}\exp\{i(a_1 X_{t_1} + \cdots + a_n X_{t_n})\} = \exp\Big\{ - \int_S \Big| \sum_{k=1}^n a_k f_{t_k}(s)\Big|^\alpha \mu(\mathrm{d}s)\Big\} \quad \forall a_i, t_i.$$

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Spectral representation by stochastic integrals

$$\{X_t\}_{t\in T} \stackrel{\mathrm{d}}{=} \Big\{\int_S f_t(s)M_\alpha(\mathrm{d}s)\Big\}_{t\in T}$$

where M_α is a SαS random measure with control measure μ.

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In short: $\{X_t\}_{t\in T} \sim \{f_t\}_{t\in T}$.

Spectral functions $\{f_t\}_{t\in T}$ not unique.

Our problem: decomposability of SαS random fields

Given SαS

$$\{X_t\}_{t\in\mathbb{Z}^d} \sim \{f_t\}_{t\in\mathbb{Z}^d}\,,$$

if $\exists$ independent SαS $\{X_t^{(k)}\}_{t\in\mathbb{Z}^d}$,

$$\{X_t\}_{t\in\mathbb{Z}^d} \stackrel{\mathrm{d}}{=} \left\{X_t^{(1)} + \cdots + X_t^{(n)}\right\}_{t\in\mathbb{Z}^d},$$

then the $\mathbf{X}^{(k)}$'s are called SαS components of $\mathbf{X}$.

- When?
- Why useful?
- Brief discussion of max-stable random fields at the end.

Decomposability

Characterization of all non-trivial (stationary) SαS $\mathbf{X}^{(k)}$'s:

$$\mathbf{X} \stackrel{\mathrm{d}}{=} \mathbf{X}^{(1)} + \cdots + \mathbf{X}^{(n)}$$

Some observations

Let $\mathbf{X}^{(k)}, k = 1, \cdots, n$ be i.i.d. copies of $\mathbf{X}$.
The sum-stability of $\mathbf{X}$ implies

$$\mathbf{X} \quad \overset{\mathrm{d}}{=} \quad n^{-\frac{1}{\alpha}}\mathbf{X}^{(1)} + \cdots + n^{-\frac{1}{\alpha}}\mathbf{X}^{(n)},$$

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$$\mathbf{X} \stackrel{\mathrm{d}}{=} n^{-\frac{1}{\alpha}}\mathbf{X}^{(1)} + \cdots + n^{-\frac{1}{\alpha}}\mathbf{X}^{(n)}, \text{ or even}$$

$$\mathbf{X} \stackrel{\mathrm{d}}{=} c_1\mathbf{X}^{(1)} + \cdots + c_n\mathbf{X}^{(n)}, \text{ for all } \sum_{i=1}^{n} |c_i|^{\alpha} = 1!$$

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- $c\mathbf{X}$ is a component of $\mathbf{X}$, for all $c \in [-1, 1]$.
- $\mathbf{X}$ has infinitely many components.

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- $c\mathbf{X}$ is a component of $\mathbf{X}$, for all $c \in [-1, 1]$.
- $\mathbf{X}$ has infinitely many components.

Focus on non-trivial SαS components: $\mathbf{X}^{(k)} \stackrel{\mathrm{d}}{\neq} c\mathbf{X}$.

Characterization of SαS components

Theorem (W, Stoev and Roy 2012)

Consider SαS random field (not necessarily stationary):

$$\{X_t\}_{t\in\mathbb{Z}^d} \sim \{f_t\}_{t\in\mathbb{Z}^d} \subset L^\alpha(S,\mu),$$

and $\mathbf{X}, \mathbf{X}^{(1)}, \dots, \mathbf{X}^{(n)}$ independent. The decomposition

$$\mathbf{X} \stackrel{\mathrm{d}}{=} \mathbf{X}^{(1)} + \cdots + \mathbf{X}^{(n)}$$

holds,

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$$\mathbf{X} \stackrel{\mathrm{d}}{=} \mathbf{X}^{(1)} + \cdots + \mathbf{X}^{(n)}$$

holds, if and only if $\exists\{r_k(s)\}_{k=1,\dots,n} \subset L^\alpha(S,\mu)$, such that

$$\{X_t^{(k)}\}_{t\in\mathbb{Z}^d} \sim \{r_k f_t\}_{t\in\mathbb{Z}^d},$$

and

$$\sum_{k=1}^n |r_k(s)|^\alpha = 1, \mu\text{-a.a. } s \in S.$$

Sketch of the proof

The 'if' part: compute characteristic function to show $\mathbf{X}^{(1)} + \cdots + \mathbf{X}^{(n)} \stackrel{\mathrm{d}}{=} \mathbf{X}$.

for all $a_1, \ldots, a_m, t_1, \ldots, t_m$ $\quad \log \mathbb{E} \exp \left\{ i \left(a_1 \sum_{k=1}^{n} X_{t_1}^{(k)} + \cdots + a_m \sum_{k=1}^{n} X_{t_m}^{(k)} \right) \right\}$

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$$\text{(by independence)} \quad = \quad \sum_{k=1}^{n} \log \mathbb{E} \exp\left\{ i\left(a_1 X_{t_1}^{(k)} + \cdots + a_m X_{t_m}^{(k)} \right) \right\}$$

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\text{(by independence)} \quad &= \sum_{k=1}^n \log \mathbb{E} \exp \left\{ i \left(a_1 X_{t_1}^{(k)} + \cdots + a_m X_{t_m}^{(k)} \right) \right\} \\
\text{(by spectral representation)} \quad &= -\sum_{k=1}^n \int_S \left| \sum_{\ell=1}^m a_\ell r_k(s) f_{t_\ell}(s) \right|^\alpha \mu(\mathrm{d}s) \\
&= -\int_S \sum_{k=1}^n |r_k(s)|^\alpha \left| \sum_{\ell=1}^m a_\ell f_{t_\ell}(s) \right|^\alpha \mu(\mathrm{d}s)
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\text{(by assumption } \textstyle\sum_{k=1}^{n} |r_k|^\alpha = 1\text{)} \quad = \quad & -\int_S \left| \sum_{\ell=1}^{m} a_\ell f_{t_\ell}(s) \right|^\alpha \mu(\mathrm{d}s)
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\end{aligned}$$

The 'only if' part: main contribution. Proof relies on minimal representations.

Example

Corollary
All SαS components of any SαS process with independent increments are still with independent increments, $\alpha \in (0, 2)$.

Result not true for Gaussian processes ($\alpha = 2$).

Stationary SαS random fields

Given a flow/$\mathbb{Z}^d$-action (Ergodic theory)

$$\{\phi_t\}_{t \in \mathbb{Z}^d} \text{ on } (S, \mu),$$

with $\phi_0(s) = s, \phi_u \circ \phi_v(s) = \phi_{u+v}(s)$,

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one obtains a stationary random field

$$\{X_t\}_{t\in\mathbb{Z}^d} \sim \{f_t\}_{t\in\mathbb{Z}^d}$$

with flow representation (Rosiński 1995, 2000)

$$f_t(s) = c_t(s) f_0 \circ \phi_t(s) \left(\frac{\mathrm{d}(\mu\circ\phi_t)}{\mathrm{d}\mu}\right)^{1/\alpha}(s), t \in \mathbb{Z}^d,$$

$\{c_t\}_{t\in\mathbb{Z}^d}$ cocycle associated to $\{\phi_t\}_{t\in\mathbb{Z}^d}$.

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$\{c_t\}_{t\in\mathbb{Z}^d}$ cocycle associated to $\{\phi_t\}_{t\in\mathbb{Z}^d}$. So

$$\{X_t\}_{t\in\mathbb{Z}^d} \sim \{f_t\}_{t\in\mathbb{Z}^d} \sim (f_0, \{\phi_t\}_{t\in\mathbb{Z}^d}).$$

Example of stationary SαS random fields

Moving averages in $\mathbb{Z}^d$:

$$X_t = \sum_{s \in \mathbb{Z}^d} a_s Z_{t-s}, t \in \mathbb{Z}^d$$

with

$$\{Z_t\}_{t \in \mathbb{Z}^d} \overset{\text{i.i.d.}}{\sim} \mathsf{S}\alpha\mathsf{S} \quad \text{and} \quad \sum_{t \in \mathbb{Z}^d} |a_t|^\alpha < \infty.$$

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(Discrete) spectral representation:

$$\{X_t\}_{t \in \mathbb{Z}^d} \sim (\Psi, \{\phi_t\}_{t \in \mathbb{Z}^d}),$$

more precisely

$$X_t = \Psi \circ \phi_t(\mathbf{Z})$$

with $\mathbb{Z}^d$-action (shift operation)

$$[\phi_t(\mathbf{z})]_s = z_{t+s}, t, s \in \mathbb{Z}^d.$$

Charaterization of stationary SαS components

Theorem (W, Stoev and Roy 2012)

For

$$\{X_t\}_{t\in\mathbb{Z}^d} \sim (f_0, \{\phi_t\}_{t\in\mathbb{Z}^d}),$$

and $\mathbf{X}, \mathbf{X}^{(1)}, \dots, \mathbf{X}^{(n)}$ independent stationary SαS. The decomposition

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holds, if and only if $\exists \{r_k(s)\}_{k=1,\dots,n}$, such that

$$\{X_t^{(k)}\}_{t\in T} \sim (r_k f_0, \{\phi_t\}_{t\in\mathbb{Z}^d}), \sum_{k=1}^n |r_k(s)|^\alpha = 1, \mu\text{-a.a. } s \in S.$$

and

$$r_k = r_k \circ \phi_t \bmod \mu, t \in \mathbb{Z}^d.$$

Example: SαS moving averages are not decomposable

Corollary

Consider SαS moving average

$$X_t = \sum_s a_s Z_{t-s}, t \in \mathbb{Z}^d.$$

If $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}$ are two independent stationary SαS random fields,

$$\mathbf{X} \stackrel{\mathrm{d}}{=} \mathbf{X}^{(1)} + \mathbf{X}^{(2)},$$

then $\mathbf{X}^{(1)} \stackrel{\mathrm{d}}{=} c\mathbf{X}$ for some constant $c \in [0, 1]$ (trivial component).

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Proof: the only invariant functions w.r.t. $\{\phi_t\}_{t\in\mathbb{Z}^d}$ are constants.$\square$

Ergodic properties

For any stationary SαS random field,

$$\{X_t\}_{t \in \mathbb{Z}^d} \stackrel{\mathrm{d}}{=} \{X_t^P + X_t^{C,N} + X_t^D\}_{t \in \mathbb{Z}^d},$$

each component have different ergodic properties, induced by group actions.

Important contributions from Cambanis, Hardin, Gross, Pipiras, Podgórski, Rosiński, Samorodnitsky, Surgailis, Taqqu, Weron, Zak, among others.

$\mathbb{Z}^d$-actions

- Given a measure space $(S, \mathcal{S}, \mu)$,

$$\phi_t : S \to S, t \in \mathbb{Z}^d$$

is a $\mathbb{Z}^d$-action on S, if

(i) $\phi_0(s) = s, s \in S$,

(ii) $\phi_{u+v}(s) = \phi_u \circ \phi_v(s), s \in S, u, v \in \mathbb{Z}^d$.

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- $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is non-singular, if $\mu \circ \phi_t \sim \mu$ for all $t \in \mathbb{Z}^d$.

- $W \in \mathcal{S}$ is a wandering set for $\{\phi_t\}_{t\in\mathbb{Z}^d}$, if

$$\{\phi_t(W)\}_{t\in\mathbb{Z}^d} \text{ are disjoint.}$$

- $W \in \mathcal{S}$ is a weakly wandering set, if $\exists\{\phi_{t_n}\}_{n\in\mathbb{N}}$ so that

$$\{\phi_{t_n}(W)\}_{n\in\mathbb{N}} \text{ are mutually disjoint mod } \mu.$$

Decompositions from ergodic theory

Conservative/dissipative decomposition (Hopf decomposition)
If $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is non-singular on S, then

$$S = C \cup D$$

with disjoint invariant measurable sets C and D, satisfying

(i) $D = \bigcup_{t\in\mathbb{Z}^d} \phi_t(W_*)$ for some wandering set W_*.

(ii) C has no wandering subset with positive measure.

Decompositions from ergodic theory

Positive/null decomposition

If $\{\phi_t\}_{t\in\mathbb{Z}^d}$ is non-singular on S, then

$$S = P \cup N$$

with disjoint invariant measurable sets P and N, satisfying

(i) $N = \bigcup_{n\in\mathbb{N}} \phi_{t_n}(W_*)$ for some weakly wandering set W_*.

(ii) P has no weakly wandering subset with positive measure.

Induced decompositions

Suppose

$$\{X_t\}_{t\in\mathbb{Z}^d} \sim (f_0, \{\phi_t\}_{t\in\mathbb{Z}^d}),$$

and $\{\phi_t\}_{t\in\mathbb{Z}^d}$ induces

$$S = C \cup D.$$

Since C and D are invariant w.r.t. $\{\phi_t\}_{t\in\mathbb{Z}^d}$,

$$\{X_t\}_{t\in\mathbb{Z}^d} \stackrel{\mathrm{d}}{=} \{X_t^C + X_t^D\}_{t\in\mathbb{Z}^d}$$

with two independent stationary SαS random fields

$$\begin{aligned} \{X_t^C\}_{t\in\mathbb{Z}^d} &\sim (f_0\mathbf{1}_C, \{\phi_t\}_{t\in\mathbb{Z}^d}) \\ \{X_t^D\}_{t\in\mathbb{Z}^d} &\sim (f_0\mathbf{1}_D, \{\phi_t\}_{t\in\mathbb{Z}^d}). \end{aligned}$$

Decomposition of stationary SαS processes/random fields

Flow representation:

$$\{X_t\}_{t\in\mathbb{Z}^d} \sim \{f_t\}_{t\in\mathbb{Z}^d} \sim \left(f_0, \{\phi_t\}_{t\in\mathbb{Z}^d}\right).$$

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Sum up:

$$\{X_t\}_{t\in\mathbb{Z}^d} \stackrel{\mathrm{d}}{=} \{X_t^P + X_t^{C,N} + X_t^D\}_{t\in\mathbb{Z}^d}.$$

Ergodic properties of stationary random fields

Given a stationary random field

$$\{X_t\}_{t\in\mathbb{Z}^d} \equiv \{X_0 \circ \theta_t\}_{t\in\mathbb{Z}^d},$$

with $\{\theta_t\}_{t\in\mathbb{Z}^d}$ measure-preserving $\mathbb{Z}^d$-action.

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The stationary random field $\{X_t\}_{t\in\mathbb{Z}^d}$ is

- ergodic, if

$$\lim_{T\to\infty} \frac{1}{T^d} \sum_{t\in\{1\ldots,T\}^d} \mathbb{P}(A\cap\theta_t(B)) = \mathbb{P}(A)\mathbb{P}(B).$$

- mixing, if

$$\lim_{n\to\infty} \mathbb{P}(A\cap\theta_{t_n}(B)) = \mathbb{P}(A)\mathbb{P}(B), \text{ for all } |t_n|\to\infty.$$

Infinitely divisible processes: ergodicity $\Leftrightarrow$ weak mixing. (Rosiński and Zak 1997).

Classification

Theorem (Rosiński, 1995 ($d = 1$), 2000 ($d > 1$))

For $\{X_t\}_{t\in\mathbb{Z}^d} \sim \{f_t\}_{t\in\mathbb{Z}^d}$, the following are equivalent:

(i) $\{X_t\}_{t\in\mathbb{Z}^d} \stackrel{\mathrm{d}}{=} \{X_t^D\}_{t\in\mathbb{Z}^d}$.

(ii) $\sum_{t\in\mathbb{Z}^d} |f_t(s)|^\alpha < \infty$ μ-a.a.

(iii) $\{X_t\}_{t\in\mathbb{Z}^d}$ is a mixed moving averages.

Mixed moving averages are mixing. (Surgailis et al. 1993).

Classification

Theorem

(Samorodnitsky, 2005 ($d = 1$), W, Roy and Stoev, 2013 ($d > 1$))

The following are equivalent:

(i) $\{X_t\}_{t \in \mathbb{Z}^d} \stackrel{\mathrm{d}}{=} \{X_t^{C,N} + X_t^D\}_{t \in \mathbb{Z}^d}$.

(ii) $\{X_t\}_{t \in \mathbb{Z}^d}$ is ergodic.

(iii) $d = 1$:

$$\sum_{t \in \mathbb{Z}} w(t) |f_t(s)|^\alpha < \infty \quad \mu\text{-a.e.}$$

for some $w(t)$ non-increasing as $t \to \pm\infty$ and

$$\sum_{t=-\infty}^{0} w(t) = \sum_{t=0}^{\infty} w(t) = \infty.$$

$d > 1$: condition of different type.

Components corresponding to different $\mathbb{Z}^d$-actions

$$\{X_t\}_{t\in\mathbb{Z}^d} \stackrel{\mathrm{d}}{=} \{X_t^P + X_t^{C,N} + X_t^D\}_{t\in\mathbb{Z}^d}.$$

$\{X_t^D\}_{t\in\mathbb{Z}^d}$: mixed moving averages, mixing.

$\{X_t^P\}_{t\in\mathbb{Z}^d}$: general form known, non-ergodic. Examples include

- Doubly stationary random fields

$$\{X_t\}_{t\in\mathbb{Z}^d} \sim \{f_t\}_{t\in\mathbb{Z}^d} \equiv \{Y_t\}_{t\in\mathbb{Z}^d} \subset L^\alpha(\Omega', \mathbb{P}')$$

  with $\{Y_t\}_{t\in\mathbb{Z}^d}$ stationary r.f. defined on $(\Omega', \mathbb{P}')$.
- Harmonizable random fields.

$\{X_t^{C,N}\}_{t\in\mathbb{Z}^d}$: few but interesting examples, ergodic.

Example from infinite ergodic theory: Poisson suspension

Let $(\Omega, \mathcal{A}, \mu, T)$ be a measure-preserving system with μ σ-finite.

Let $\mathcal{P}$ be the Poisson random measure on Ω with intensity μ.

Counting measure of $A \in \mathcal{A}$:

$$\frac{1}{n} S_n(A) = \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{P}(T^{-k} A) \to \mu(A), \text{ almost surely.}$$

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Under suitable condition,

$$\frac{S_n(A) - n\mu(A)}{\sigma_n(A)} \Rightarrow \mathcal{N}(0, 1).$$

Situation complicated when μ is infinite and T is conservative-null (Zweimüller, 2008).

Consequences

Given $\{X_t\}_{t \in \mathbb{Z}^d}$, what is the corresponding $\{\phi_t\}_{t \in \mathbb{Z}^d}$?

- Better understanding of known random fields.

Given $\{\phi_t\}_{t \in \mathbb{Z}^d}$, what is the corresponding $\{X_t\}_{t \in \mathbb{Z}^d}$?

- Different types of dependence.
- Different asymptotic behaviors.
- Construction for new random fields.

An extremal limit theorem

Theorem (Roy and Samorodnitsky, 2008)
For a stationary SαS random field $\{X_t\}_{t\in\mathbb{Z}^d} \sim (f_0, \{\phi_t\}_{t\in\mathbb{Z}^d})$, write

$$M_n = \max_{i\in\{1,\dots,n\}^d} |X_i|.$$

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- If $\{X_t^D\}_{t\in\mathbb{Z}^d}$ is non-degenerate, then

$$\frac{1}{n^{d/\alpha}} M_n \Rightarrow c_{\alpha,f_0} Z_\alpha$$

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- If $\{X_t^D\}_{t\in\mathbb{Z}^d}$ is degenerate, then

$$\frac{1}{n^{d/\alpha}} M_n \to 0 \text{ in probability.}$$

Different flows generate long/short range dependence.

A central limit theorem

β-Mittage–Leffler fractional SαS motion
(Owada and Samorodnitsky, 2013)

- $\{X_n\}_{n\in\mathbb{N}}$ a stationary infinitely divisible process.
- driven by a conservative-null flow.
- Invariance principle

$$\left\{\frac{1}{c_n}\sum_{i=1}^{\lfloor nt\rfloor} X_i\right\}_{t\in[0,1]} \Rightarrow \{Y_{\alpha,\beta}(t)\}_{t\in[0,1]}.$$

- β: speed of null recurrence in terms of pointwise dual ergodic theory.
- $Y_{\alpha,\beta}$ includes limiting processes arising from random walks in random scenery (Dombry and Guillotin–Plantard, 2009).

A few words on max-stable random fields

Max-stable distributions/random fields

Y is max-stable, if

$$a_n Y + b_n \stackrel{\mathrm{d}}{=} \max_{i=1,\ldots,n} Y_i .$$

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Three types: Fréchet, Gumbel and negative Fréchet.

Fréchet distribution: $\mathbb{P}(Y \leq y) = \exp\left(-y^{-\alpha}\right), y \geq 0$:

$$Y \stackrel{\mathrm{d}}{=} 2^{-1/\alpha} \max\{Y_1, Y_2\}.$$

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$\{Y_t\}_{\mathbb{Z}^d}$ is an α-stable random field, if

$$\max_{i=1,\dots,n} a_i Y_{t_i} \quad \text{is } \alpha\text{-Fréchet, for all } a_i > 0, t_i.$$

Representation of α-Fréchet random fields

Extremal stochastic integral representation (Stoev and Taqqu, 2006)

$$Y_t = \int_S^{\vee} f_t(s) M_\alpha^{\vee}(\mathrm{d}s), t \in \mathbb{Z}^d,$$

where $\{f_t\}_{t\in\mathbb{Z}^d} \subset L_+^\alpha(S,\mu)$, and $M_\alpha^\vee$ is an α-Fréchet random sup-measure.

Finite-dimensional distribution: for all $a_i > 0, t_i$,

$$\mathbb{P}(Y_{t_i} \le a_i, i = 1, \dots, n) = \exp\left\{ - \int_S \max_{i=1,\dots,n} \left(\frac{f_{t_i}(s)}{a_i} \right)^\alpha \mu(\mathrm{d}s) \right\}.$$

Max-stable random fields

In short,

$$\{Y_t\}_{t\in\mathbb{Z}^d} \sim \{f_t\}_{t\in\mathbb{Z}^d} \subset L_+^\alpha(S,\mu).$$

Max-stable random fields can be associated to SαS random fields with the same spectral functions. The inverse not true.
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Max-decomposability

$$\{Y_t\}_{t\in\mathbb{Z}^d} \stackrel{\mathrm{d}}{=} \{Y_t^{(1)} \vee Y_t^{(2)}\}_{t\in\mathbb{Z}^d}.$$

Summary

Characterization of all SαS components.

Close connections between stationary SαS random fields and group actions.

Extension from stochastic processes to random fields not simple: tools from ergodic theory for $d = 1$ and $d > 1$ may be different.

Examples from ergodic theory for different dependence structures.

Similar situations/results for max-stable random fields.

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