FRACTIONAL POISSON FIELD ON A FINITE SET

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Abstract. The fractional Poisson field (fPf) can be interpreted in term of the number of balls falling down on each point of $\mathbb{R}^D$, when the centers and the radii of the balls are thrown at random following a Poisson point process in $\mathbb{R}^D \times \mathbb{R}^+$ with an appropriate intensity measure. It provides a simple description for a non Gaussian random field that has the same covariance function as the fractional Brownian field. In the present paper, we concentrate on the restrictions of the fPf to finite sets of points in $\mathbb{R}^D$. Actually, since it takes discrete values, it seems natural to adapt this field to a discrete context. We are particularly interested in its finite-dimensional distributions, in its representation on a finite grid, and in its discrete variations which yield an estimator for its Hurst index.

Introduction

In the last decades a lot of papers have been dedicated to the sum of an infinite number of Poisson sources. The seminal ideas of Mandelbrot of adding Poisson sources in order to get a fractional limit are described for instance in [5]. More recently this subject became popular for the modeling of Internet traffic and telecommunication (see [6, 15]) providing processes with heavy tails or long range dependence. In higher dimension, throwing Euclidean balls at random following a specific Poisson repartition for the centers and the radii, and counting how many balls fall down on each point, provides a random field defined on $\mathbb{R}^D$. In [13], with an appropriate scaling, a generalized random field is obtained as an asymptotics. It has a Poisson structure and exhibits a kind of self-similarity index H greater than $1/2$. The case H less than $1/2$ is studied in [2] and a pointwise representation $(F_H(y))_{y\in\mathbb{R}^D}$ of the generalized field is given. It is proved that F_H may be written as an integral with respect to a Poisson random measure and F_H is called as *fractional Poisson field* (fPf). Actually the fPf is of own interest since it provides a microscopic description of macroscopic properties like stationary increments or self-similarity and since it has the same covariance function as the fractional Brownian field but is not Gaussian. Consequently, one can get the fractional Brownian field with a central limit theorem procedure starting from copies of the fPf. For such an approach see [10]. Moreover let us mention the opportunity of obtaining many other models following the same scheme. For instance one can build anisotropic fields by replacing the Euclidean balls by more general convex sets, stable fields by adding weights to balls [3] and natural images can be simulated [4].

The present paper focuses on the restrictions of the fPf to finite sets of points in $\mathbb{R}^D$. It is organized as follows. In the first section, we concentrate on the finite-dimensional distributions of the fPf. Meanwhile, we exhibit a representation of F_H similar to the Chentsov one (see [19], Chapter 8). In particular, we establish that all the finite dimensional distributions are determined by the $(D+1)$-dimensional marginal distributions. In Section 2, we give a

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representation of the fPf on a finite grid $\Gamma \subset \mathbb{R}^D$. More precisely, using explicit random variables, we describe a random vector $(G(y))_{y\in\Gamma}$ that has the same distribution as $(F_H(y))_{y\in\Gamma}$. Such a representation could be used for simulation purposes. In the last section, namely Section 3, we investigate the increments of a fPf. We first give an estimate of the expectation of the q-structure functions of F_H for a step δ and establish that it behaves as δ^{2H} with a constant power, pointing out a high irregularity of F_H. We then turn to the discrete quadratic variations of the fPf. A ratio of two different quadratic variations yields an a.s. estimator for the Hurst index H. Note that a similar result holds for the fractional Brownian field, but that our proof needs new arguments since we don't deal anymore with a Gaussian framework.

To end this section let us give the notations used in the sequel and the precise definition of the fPf as introduced in [2].

We consider $\mathbb{R}^D$ endowed with the Euclidean norm $\|\cdot\|$. We write $B(x,r)$ for the closed ball of center x and radius $r>0$ with respect to the Euclidean norm. Without any risk of confusion, the notation $|\cdot|$ will either denote the absolute value of any real number, or the D-dimensional Lebesgue measure of any measurable subset of $\mathbb{R}^D$. In what follows, we will write V_D for $|B(0,1)|$, the volume of the unit Euclidean ball in $\mathbb{R}^D$, and S^{D-1} for the unit sphere in $\mathbb{R}^D$.

Let $H\in(0,1/2)$. We consider Φ_H a Poisson point process in $\mathbb{R}^D\times\mathbb{R}^+$ with intensity

$$\nu_H(dx,dr)=r^{-D-1+2H}\,dx\,dr, \tag{1}$$

and associate with Φ_H a Poisson random measure N_H on $\mathbb{R}^D\times\mathbb{R}^+$ with the same intensity measure ν_H.

For any y in $\mathbb{R}^D$, we consider the stochastic integral

$$F_H(y)=\int_{\mathbb{R}^D\times\mathbb{R}^+}\left(\mathbb{1}_{B(x,r)}(y)-\mathbb{1}_{B(x,r)}(0)\right)\,N_H(dx,dr) \tag{2}$$

and finally we introduce the *fractional Poisson field with Hurst index H* as the random field $F_H=(F_H(y))_{y\in\mathbb{R}^D}$, which is clearly centered with stationary increments.

Heuristically, $F_H(y)$ may be seen as the difference between the number of balls $B(x,r)$ with $(x,r)\in\Phi_H$ covering the point y, and the number of balls covering the origin. This point of view is not fair since the number of balls covering one particular point is infinite. Nevertheless, the stochastic integral (2) is well defined since $\mathbb{1}_{B(x,r)}(y)-\mathbb{1}_{B(x,r)}(0)$ belongs to $L^1(\mathbb{R}^D\times\mathbb{R}^+,\nu_H(dx,dr))$. Actually, for any $y\in\mathbb{R}^D$, one can find a constant $C(y)\in(0,+\infty)$ such that for any $r\in\mathbb{R}^+$,

$$\int_{\mathbb{R}^D}|\mathbb{1}_{B(x,r)}(y)-\mathbb{1}_{B(x,r)}(0)|\,dx=|B(y,r)\triangle B(0,r)|\leqslant C(y)\min\left(r^D,r^{D-1}\right) \tag{3}$$

where $A\triangle B$ stands for the symmetric difference between A and B, two subsets of $\mathbb{R}^D$.

Furthermore, for any $y\in\mathbb{R}^D$, $\mathbb{1}_{B(x,r)}(y)-\mathbb{1}_{B(x,r)}(0)$ also belongs to $L^2(\mathbb{R}^D\times\mathbb{R}^+,\nu_H(dx,dr))$ and using the rotation invariance of the Lebesgue measure, we obtain

$$\int_{\mathbb{R}^D\times\mathbb{R}^+}\left(\mathbb{1}_{B(x,r)}(y)-\mathbb{1}_{B(x,r)}(0)\right)^2\,\nu_H(dx,dr)=c_H\,\|y\|^{2H} \tag{4}$$

with

$$c_H=\int_{\mathbb{R}^+}|B(e_1,r)\triangle B(0,r)|\,r^{-D-1+2H}dr$$

and e_1 being any point in S^{D-1}. One can check that c_H is explicitly computed in dimension $D=1$ as $c_H=\frac{2^{1-2H}}{H(1-2H)}$. In higher dimension, explicit formulas for $|B(e_1,r)\triangle B(0,r)|$ can be found for instance in [20].

Equation (4) shows that, up to a multiplicative constant, F_H has the same covariance function as B_H, the fractional Brownian field (fBf) of Hurst index H, namely

$$\mathrm{Cov}(F_H(y), F_H(y')) = c_H \ \mathrm{Cov}(B_H(y), B_H(y')) = \frac{1}{2} c_H \left(\|y\|^{2H} + \|y'\|^{2H} - \|y - y'\|^{2H} \right). \quad (5)$$

Using a Gaussian measure with control measure ν_H instead of the Poisson measure N_H in (2), would provide a Gaussian field with covariance (5) that would actually, up to a constant, be the fBf of index H. Contrarily to this last one, the fPf is neither Gaussian nor self-similar. However it is still with stationary increments, it is second-order self-similar and it exhibits what is called an aggregate similarity property in [14].

Since the value of H is fixed all over the paper, we will not mention the dependence on H anymore and from now on we will drop all the H indices writing Φ, N, ν, F instead of Φ_H, N_H, ν_H, F_H.

1. Chentsov representation

We notice that for $x, y \in \mathbb{R}^D$ and $r \in \mathbb{R}^+$ we have

$$\mathbb{1}_{B(x,r)}(y) - \mathbb{1}_{B(x,r)}(0) = \mathbb{1}_{\mathcal{C}(y) \cap \mathcal{C}(0)^c}(x, r) - \mathbb{1}_{\mathcal{C}(0) \cap \mathcal{C}(y)^c}(x, r) \ ,$$

when defining $\mathcal{C}(y)$, the *cone over* y, by

$$\mathcal{C}(y) = \{(x, r) \in \mathbb{R}^D \times \mathbb{R}^+ \, ; \, y \in B(x, r)\} \, . \quad (6)$$

A similar computation as the one in (4) gives

$$\nu(\mathcal{C}(y) \cap \mathcal{C}(0)^c) = \int_{\mathbb{R}^D \times \mathbb{R}^+} \mathbb{1}_{B(y,r) \cap B(0,r)^c}(x) \, \nu(dx, dr) = \frac{c_H}{2} \|y\|^{2H} \, .$$

Then, we can write

$$F(y) = N(\mathcal{C}(y) \cap \mathcal{C}(0)^c) - N(\mathcal{C}(0) \cap \mathcal{C}(y)^c) \quad (7)$$

and observe that $F(y)$ follows a Skellam distribution: it is equal to the difference of two i.i.d. Poisson random variables with parameter $\frac{c_H}{2} \|y\|^{2H}$.

This formulation invites us to link the fPf to another related fields G which can be written as

$$G(y) = M(\mathcal{C}(y) \cap \mathcal{C}(0)^c) - M(\mathcal{C}(0) \cap \mathcal{C}(y)^c) \quad (8)$$

where M is any random measure on $\mathbb{R}^D$ such that (8) makes sense and $\mathcal{C}(y)$ is the cone over y as in (6).

First, when M is replaced by a symmetric α-stable random measure, the resulting field is a so-called "$H - sssis$ (H self-similar with stationary increments in the strong sense) $S\alpha S$ Chentsov field" as introduced in [19], with the resulting consequence that $H \leqslant 1/\alpha$. Going further, M still being replaced by a symmetric α-stable random measure, and replacing the difference in (8) by the sum, then the resulting field would be a Takenaka random field [21].

Actually, for the Takenaka random fields, as well as for the fields defined by (8), the following proposition holds.

Proposition 1.1. *Let G be defined by* (8). *Let $y_1, y_2, \ldots, y_m$ be m points in $\mathbb{R}^D \setminus \{0\}$ with $m > D$. Then the distribution of $(G(y_1), \ldots, G(y_m))$ is determined by the $(D+1)$-dimensional marginal distributions of G.*

A similar result was originally established by Sato in [18] for Takenaka fields. As a consequence of Proposition 1.1, if a field G associated with an unknown Poisson measure M has the same $(D+1)$-dimensional marginal distributions as the fPf F, then realizations of G may be obtained by choosing M as the particular Poisson measure with intensity (1).

We do not detail the proof of Proposition 1.1 since similar ideas as [18] can be used in our case. We only describe the ingredients which are needed.

For any positive integer m, we define

$$\mathcal{E}_m = \{0,1\}^m = \{e : [\![1,m]\!] \to \{0,1\}\} \ .$$

Let $y_1, y_2, \dots, y_m$ be fixed in $\mathbb{R}^D \setminus \{0\}$. Then, writing $T = (y_1, y_2, \dots, y_m)$ we denote for any $e \in \mathcal{E}_m$,

$$C(T,e) = \bigcap_{1 \leqslant k \leqslant m} \mathcal{C}(y_k)^{e(k)}$$

where $\mathcal{C}(y)$ still stands for the cone over y and the following convention is used

$$\mathcal{C}(y)^1 = \mathcal{C}(y) \ \text{ and } \ \mathcal{C}(y)^0 = \mathcal{C}(y)^c \ .$$

The next statements are obvious. For $e, e' \in \mathcal{E}_m$, if $e \neq e'$ then $C(T,e) \cap C(T,e') = \emptyset$, and for any $k = 1, \dots, m$,

$$\mathcal{C}(y_k) = \bigcup_{e \in \mathcal{E}_m ;\, e(k)=1} C(T,e) \ \text{ and } \ \mathcal{C}(y_k)^c = \bigcup_{e \in \mathcal{E}_m ;\, e(k)=0} C(T,e) \ . \tag{9}$$

We also denote $\mathring{T} = (0, y_1, y_2, \dots, y_m)$ and $\mathring{\mathcal{E}}_m = \{e : [\![0,m]\!] \to \{0,1\}\}$, so that using (9), for any $k = 1, \dots, m$,

$$\begin{aligned} \mathcal{C}(0)^c \cap \mathcal{C}(y_k) &= \bigcup_{e \in \mathring{\mathcal{E}}_m ;\, e(0)=0, e(k)=1} C(\mathring{T}, e) \ , \\ \mathcal{C}(0) \cap \mathcal{C}(y_k)^c &= \bigcup_{e \in \mathring{\mathcal{E}}_m ;\, e(0)=1, e(k)=0} C(\mathring{T}, e) \ . \end{aligned}$$

Hence, using (7), we obtain a representation of the random vector $(F(y_1), \dots, F(y_m))$ as stated in the next proposition.

Proposition 1.2. *Let $y_1, y_2, \dots, y_m \in \mathbb{R}^D \setminus \{0\}$ and $\mathring{\mathcal{E}}_m = \{e : [\![0,m]\!] \to \{0,1\}\}$. There exists a family of independent Poisson random variables $\{X(e) ; e \in \mathring{\mathcal{E}}_m\}$ such that*

$$\big(F(y_k)\big)_{1 \leqslant k \leqslant m} = \Bigg(\sum_{e \in \mathring{\mathcal{E}}_m} (e(k) - e(0)) X(e) \Bigg)_{1 \leqslant k \leqslant m} . \tag{10}$$

Moreover, for any $e \in \mathring{\mathcal{E}}_m$, $X(e) = N(C(\mathring{T}, e))$.

2. Discrete representation

Let us fix $0 < \delta < R$ and consider the finite set of $\mathbb{R}^D$ with $J_{R,\delta} \in \mathbb{N}$ points

$$\Gamma_{R,\delta} = B(0,R) \cap \delta \mathbb{Z}^D = \{y_j \, ; \, 1 \leqslant j \leqslant J_{R,\delta}\}. \tag{11}$$

We discuss here the possibility to represent the discrete field $(F(y))_{y \in \Gamma_{R,\delta}}$ by a simpler and 'more natural' field which could be more relevant for the structure of F. The idea is to count the number of balls $B(x,r)$ falling down on the points of $\Gamma_{R,\delta}$. For any fixed $y \in \mathbb{R}^D$, we take into account in the integral (2) both the covering of y and 0 because the function $(x,r) \mapsto \mathrm{I}_{B(x,r)}(y)$ is not integrable with respect to the intensity measure ν given by (1). Intuitively the fact that this latter integral is not finite comes from the high number of very large balls. It is possible to classify the balls according to their influence on the finite set $\Gamma_{R,\delta}$. Notice that $0 \in \Gamma_{R,\delta}$. One checks that, for all $y \in \Gamma_{R,\delta}$,

- if $r + R < \|x\|$ then $B(x,r)$ does not intersect $B(0,R)$ so $\mathbb{1}_{B(x,r)}(y) = \mathbb{1}_{B(x,r)}(0) = 0$,
- if $\|x\| + R \leqslant r$ then $B(x,r)$ covers $B(0,R)$ so $\mathbb{1}_{B(x,r)}(y) = \mathbb{1}_{B(x,r)}(0) = 1$,
- if $(\|x\| - R)_+ \leqslant r < \|x\| + R$ then $B(x,r)$ does not cover $B(0,R)$ but is with a non empty intersection with $B(0,R)$ so $\mathbb{1}_{B(x,r)}(y) - \mathbb{1}_{B(x,r)}(0) \in \{-1,0,1\}$.

Each type of balls corresponds to a Poisson Point process (PPP) with a suitable intensity and by superposition, the original PPP Φ corresponds to their independent union. Only the balls that have a non-trivial intersection with $B(0,R)$ are interesting. They are related to a PPP in $\mathbb{R}^D \times \mathbb{R}^+$ of intensity measure

$$\nu_0(dx,dr) = \mathbb{1}_{[(\|x\|-R)_+,\|x\|+R)}(r) r^{-D-1+2H}\, dx\, dr.$$

In order to deal with the balls with small radii (smaller than $\delta/2$) we use independence and superposition property by splitting the intensity ν_0 as $\nu^{(1)} + \nu^{(2)}$ with

$$\nu^{(1)}(dx,dr) = \mathbb{1}_{[(\|x\|-R)_+,\|x\|+R)\cap[\delta/2,+\infty)}(r) r^{-D-1+2H}\, dx\, dr$$
$$\nu^{(2)}(dx,dr) = \mathbb{1}_{[(\|x\|-R)_+,\|x\|+R)\cap[0,\delta/2)}(r) r^{-D-1+2H}\, dx\, dr.$$

Balls with large radii. Let us consider a global PPP $\Phi^{(1)}$ of intensity $\nu^{(1)}$. The number of associated balls is a.s. finite and Poisson distributed with parameter

$$\lambda_1 = \int_{\mathbb{R}^D \times \mathbb{R}^+} \nu^{(1)}(dx,dr) = \int_{\delta/2}^{+\infty} C_1(r) r^{-D-1+2H} dr$$

with

$$C_1(r) = \int_{\mathbb{R}^D} \mathbb{1}_{[(r-R)_+,r+R]}(\|x\|) dx = V_D \left((r+R)^D - (r-R)_+^D \right).$$

Note that, since R is fixed, as r tends to infinity, $C_1(r) r^{-D-1+2H}$ behaves like r^{-2+2H}. Hence, since $H < 1/2$, the last integral converges and $\lambda_1 < \infty$. Therefore we can decompose the intensity $\nu^{(1)}(dx,dr)$ as

$$\underbrace{\lambda_1}_{\substack{\text{number of balls}\\ \text{to consider}}} \underbrace{\left(\frac{C_1(r)}{\lambda_1} r^{-D-1+2H} \mathbb{1}_{[\delta/2,+\infty)}(r) dr \right)}_{\text{distribution of the large radii}} \underbrace{\left(\frac{1}{C_1(r)} \mathbb{1}_{[(r-R)_+,r+R]}(\|x\|) dx \right)}_{\substack{\text{distribution of the centers}\\ \text{conditionally to the radii}}}. \tag{12}$$

Thus we define a random field $T^{(1)}$ by

$$T^{(1)} = \sum_{n=1}^{\Lambda_1} \mathbb{1}_{B(X_n^{(1)}, R_n^{(1)})} \tag{13}$$

where

- Λ_1 is a Poisson random variable with parameter λ_1
- $R_n^{(1)}$ is a positive random variable with probability density function

$$\rho_1(r) = \lambda_1^{-1} C_1(r)\, r^{-D-1+2H} \mathbb{1}_{[\delta/2,+\infty)}(r)$$

- $X_n^{(1)}$ is distributed in $\mathbb{R}^D$ according to the probability distribution with conditional density with respect to $R_n^{(1)}$ given by

$$\gamma_1^{\{R_n^{(1)}=r\}}(x) = \frac{1}{C_1(r)} \mathbb{1}_{[(r-R)_+, r+R]}(\|x\|).$$

Balls with small radii. Now we focus on the intensity measure $\nu^{(2)}(dx, dr)$. Let $(x, r) \in \mathbb{R}^D \times \mathbb{R}^+$. Either $(x, r) \in \bigcap_{j=1}^{J_{\delta,R}} \mathcal{C}(y_j)^c$ and the ball $B(x, r)$ have no contribution on the set $\Gamma_{R,\delta}$, either $(x, r) \in \bigcup_{j=1}^{J_{\delta,R}} \mathcal{C}(y_j)$. Since $\|y_j - y_i\| \geqslant \delta$ for all pairs (y_i, y_j) of different points in $\Gamma_{R,\delta}$, the $J_{R,\delta}$ sets $(\mathbb{R}^D \times [0, \delta/2)) \cap \mathcal{C}(y_j)$ are disjoint sets. Therefore the PPP of intensity $\nu^{(2)}$ is the superposition of $J_{R,\delta}$ independent PPP $\Phi_j^{(2)}$ of intensity

$$\nu_j^{(2)}(dx, dr) = \mathbb{1}_{(\mathbb{R}^D \times [\|x\|-R, \|x\|+R] \cap [0,\delta/2)) \cap \mathcal{C}(y_j)}(x, r) r^{-D-1+2H}\, dx\, dr$$

for which the associated balls $B(x, r)$ satisfy $\mathbb{1}_{B(x,r)}(y_i) = 1$ if and only if $i = j$.
Thus each PPP provides a random field $T_j^{(2)}$ such that

$$T_j^{(2)}(y_i) = \begin{cases} \Lambda_j^{(2)} & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

where $\Lambda_j^{(2)}$ is the number of balls resulting from $\Phi_j^{(2)}$. This number is a.s. finite and Poisson distributed according to the parameter $\lambda_j^{(2)} = \nu_j^{(2)}(\mathbb{R}^D \times \mathbb{R}^+)$. Since $\|x - y_j\| \geqslant \|x\| - \|y_j\| \geqslant \|x\| - R$ and $\|x\| + R \geqslant R > \delta/2$, we obtain

$$\lambda_j^{(2)} = \int_{\mathbb{R}^D \times \mathbb{R}^+} \mathbb{1}_{[\|x-y_j\|, \delta/2)}(r) r^{-D-1+2H} dx\, dr = \frac{V_D}{2H}(\delta/2)^{2H}.$$

To conclude we define a random field $T^{(2)}$ over $\Gamma_{R,\delta}$ by $T^{(2)} = \sum_{j=1}^{J_{R,\delta}} T_j^{(2)}$ (note that the $T_j^{(2)}$ are independent). Finally, superposing all the previous independent PPP and their related fields, we obtain the following proposition.

Proposition 2.1. *Let $\Gamma_{R,\delta}$ be the finite set defined by* (11)*. Then $(F(y))_{y \in \Gamma_{R,\delta}}$ has the same distribution as $(G(y) - G(0))_{y \in \Gamma_{R,\delta}}$ with $G = T^{(1)} + T^{(2)}$.*

This description shows that the restriction to $\Gamma_{R,\delta}$ of the field F is essentially made up with

- a field $T^{(1)}$ which is a simple 'balls counting field': random balls are built picking-up the radii first, the centers next, then $T^{(1)}(y_j)$ counts the number of these balls above each y_j,
- a field $T^{(2)}$ whose values at each point y_j form a collection of i.i.d. Poisson random variables with parameter $\frac{V_D}{2H}(\delta/2)^{2H}$.

3. Increments of a fPf

3.1. Structure functions.

Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. If $q > 0$, we define the *q-structure function* of f by

$$\forall\, \delta \geqslant 0 \quad S_q(\delta) = \int_0^1 |f(t+\delta) - f(t)|^q \, dt. \tag{14}$$

When f is regular enough one may expect that $\lim_{\delta \to 0} S_q(\delta) = 0$ for all $q > 0$. More precisely if f satisfies $|f(t+\delta) - f(\delta)| \simeq_0 \delta^\alpha$ with $\alpha \in (0,1]$ and for all $t \in \mathbb{R}$ (where $u(\delta) \simeq_0 v(\delta)$ means that $u(\delta)/v(\delta)$ is bounded from above and from below for small δ) then clearly $S_q(\delta) \simeq_0 \delta^{\alpha q}$.

When f is not as regular these functionals are a classical tool to study its fractal behavior [12, 22]. In view of the regular case, one is usually interested in an asymptotic power-law behavior through a relation of the type $S_q(\delta) \simeq_0 \delta^{H(q)}$ for a certain constant $H(q) > 0$.

The study of random processes forces to deal with sample paths and the relation above has to hold almost surely. For example for the fractional Brownian paths B_H one obtains $S_q(\delta) \simeq_0 \delta^{2Hq}$ and $H(q) = 2Hq$ with probability one. Except for a few random functions such computations are difficult to state. Instead of working path-to-path one may prefer to work statistically (see [7, 16]) looking for relations of the type

$$\mathbb{E}(S_q(\delta)) \simeq_0 \delta^{H(q)}. \tag{15}$$

Then, assuming that f is a random function satisfying $\sup_{t \in [-s, 1+s]} \mathbb{E}\, |f(t)|^q < \infty$ for some $s > 0$, one can consider, by Fubini theorem,

$$\mathbb{E}(S_q(\delta)) = \int_0^1 \mathbb{E}\, |f(t+\delta) - f(t)|^q \, dt. \tag{16}$$

Finally, let us explain how to work with a multivariate function $f : \mathbb{R}^D \to \mathbb{R}$. Among all the possible extensions we choose to look at the function along straight lines: for fixed $t_0 \in \mathbb{R}^D$ and direction $\theta \in S^{D-1}$ we define

$$t \in \mathbb{R} \mapsto L_{t_0,\theta} f(t) = f(t_0 + t\theta)\,.$$

Then one may consider (16) for the univariate functions $L_{t_0,\theta} f$.

Now we can state the main result of this section: for the fPf F, all the functions $L_{t_0,\theta} F$ satisfy the relation (15) for all $q \geqslant 2$ and $H(q)$ is explicit.

Theorem 3.1. *For all $q \geqslant 2$, there exists C_q, C'_q, $\delta_q > 0$ such that, for all $t_0 \in \mathbb{R}^D$ and all $\theta \in S^{D-1}$ one has*

$$\forall\, \delta \in [0, \delta_q] \qquad C_q\, \delta^{2H} \leqslant \mathbb{E}\left(\int_0^1 |F(t_0 + (t+\delta)\theta) - F(t_0 + t\theta)|^q \, dt \right) \leqslant C'_q\, \delta^{2H}. \tag{17}$$

Proof. The proof is divided into two parts: (i) the result is proved for all even integers, (ii) we extend it to all integers q using Hölder interpolation.

Let $t_0 \in \mathbb{R}^D$ and $\theta \in S^{D-1}$. For all $t \in \mathbb{R}$ we simply write $f(t) = F(t_0 + t\theta)$.

Let us write, for all $(y, x, r) \in \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{R}^+$,

$$\psi(y, x, r) = \mathbb{1}_{B(x,r)}(y) - \mathbb{1}_{B(x,r)}(0).$$

First we observe that for all $t \in \mathbb{R}$ and $\delta \geqslant 0$:

$$f(t+\delta) - f(t) = \int_{\mathbb{R}^D \times \mathbb{R}^+} \left(\psi(t_0 + (t+\delta)\theta, x, r) - \psi(t_0 + t\theta, x, r)\right) N(dx, dr)$$
$$= \int_{\mathbb{R}^D \times \mathbb{R}^+} \psi(\delta\theta, x - t_0 - t\theta, r) N(dx, dr) \,. \quad (18)$$

(i) Let $p \geqslant 1$. Observe now that $\psi(y, \cdot, \cdot) \in L^{2p}\left(\mathbb{R}^D \times \mathbb{R}^+, \nu(dx, dr)\right)$ and

$$\int_{\mathbb{R}^D \times \mathbb{R}^+} \psi(\delta\theta, x - t_0 - t\theta, r)^p \, \nu(dx, dr) = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ c_H \delta^{2H} & \text{if } p \text{ is even.} \end{cases}$$

Then, according to [1] (with the convention that $0^0 = 1$) and using (18), we have

$$\mathbb{E}\left((f(t+\delta) - f(t))^{2p}\right)$$
$$= \sum_{(r_1, \ldots, r_{2p}) \in I(2p)} K_{2p}(r_1, \ldots, r_{2p}) \prod_{k=1}^{2p} \left(\int_{\mathbb{R}^D \times \mathbb{R}^+} \psi(\delta\theta, x - t_0 - t\theta, r)^k \, \nu(dx, dr)\right)^{r_k}$$
$$= \sum_{(0, r_2, 0, \ldots, r_{2p}) \in I(2p)} K_{2p}(0, r_2, 0, \ldots, r_{2p}) \prod_{k=1}^{p} \left(\int_{\mathbb{R}^D \times \mathbb{R}^+} \psi(\delta\theta, x - t_0 - t\theta, r)^{2k} \, \nu(dx, dr)\right)^{r_{2k}}$$
$$= \sum_{(r_1, \ldots, r_p) \in I(p)} \widetilde{K}_p(r_1, \ldots, r_p) \left(c_H \delta^{2H}\right)^{\sum_{k=1}^{p} r_k} \,,$$

where, for $n \geqslant 1$, $I(n) = \left\{(r_1, \ldots, r_n) \in \mathbb{N}^n; \sum_{k=1}^{n} k r_k = n\right\}$, $K_n(r_1, \ldots, r_n) = n! \left(\prod_{k=1}^{n} r_k! (k!)^{r_k}\right)^{-1}$ and $\widetilde{K}_n(r_1, \ldots, r_n) = (2n)! \left(\prod_{k=1}^{n} r_k! ((2k)!)^{r_k}\right)^{-1}$. Integrating with respect to $t \in [0,1]$ we get

$$\mathbb{E}\left(S_{2p}(\delta)\right) = \sum_{(r_1, \ldots, r_p) \in I(p)} \widetilde{K}_p(r_1, \ldots, r_p) \left(c_H \delta^{2H}\right)^{\sum_{k=1}^{p} r_k} .$$

Note that $e_p = (0, \ldots, 0, 1) \in I(p)$ and for any $(r_1, \ldots, r_p) \in I(p) \setminus \{e_p\}$ we have $\sum_{k=1}^{p} r_k \geqslant 2$ such that

$$\mathbb{E}\left(S_{2p}(\delta)\right) = c_H \delta^{2H} + \delta^{4H} u(\delta) \,,$$

where $\delta \mapsto u(\delta)$ is bounded near 0. This gives the result for all even q.

(ii) We will prove that for all $q \geqslant 2$, there exists $C_q, C'_q, \delta_q > 0$ such that

$$\forall\, t \in [0,1] \quad \forall\, \delta \in [0, \delta_q] \quad C_q\, \delta^{2H} \leqslant \mathbb{E}\left|f(t+\delta) - f(t)\right|^q \leqslant C'_q\, \delta^{2H}. \quad (19)$$

Let $t \in [0,1]$ and $\delta \geqslant 0$.

(a) Let $1 \leqslant q \leqslant r \leqslant q' < +\infty$ and $\alpha \in [0,1]$ such that $\frac{1}{r} = \frac{\alpha}{q} + \frac{1-\alpha}{q'}$.

Then, using Hölder inequality :

$$\mathbb{E}\left|f(t+\delta) - f(t)\right|^r \leqslant \left(\mathbb{E}\left|f(t+\delta) - f(t)\right|^q\right)^{\frac{\alpha r}{q}} \left(\mathbb{E}\left|f(t+\delta) - f(t)\right|^{q'}\right)^{\frac{(1-\alpha) r}{q'}}. \quad (20)$$

(b) *Proof of the rhs of* (19). Let $r \in [2, +\infty)$ and $p \in \mathbb{N} \setminus \{0\}$. Apply (20) with $q = 2p$ and $q' = 2p + 2$ yields

$$\mathbb{E}\,|f(t+\delta) - f(t)|^r \leqslant (\mathbb{E}\,|f(t+\delta) - f(t)|^{2p})^{\frac{\alpha r}{2p}} (\mathbb{E}\,|f(t+\delta) - f(t)|^{2p+2})^{\frac{(1-\alpha)r}{2p+2}}.$$

Therefore, using (i):

$$\mathbb{E}\,|f(t+\delta) - f(t)|^r \leqslant (C'_{2p} C'_{2p+2}) \delta^{(2H)(\frac{\alpha r}{2p} + \frac{(1-\alpha)r}{2p+2})} = C'_r \delta^{2H}$$

assuming $0 \leqslant \delta < \min(\delta_{2p}, \delta_{2p+2})$.

(c) *Proof of the lhs of* (19). Let $r \in [1, +\infty)$ and $p \in \mathbb{N} \setminus \{0\}$ such that $1 \leqslant r \leqslant 2p \leqslant 2p + 2$. Apply (20) with $q = r$, $r = 2p$ and $q' = 2p + 2$ yields

$$\mathbb{E}\,|f(t+\delta) - f(t)|^{2p} \leqslant (\mathbb{E}\,|f(t+\delta) - f(t)|^r)^{\frac{2p\,\alpha}{r}} (\mathbb{E}\,|f(t+\delta) - f(t)|^{2p+2})^{\frac{(1-\alpha)2p}{2p+2}}.$$

Therefore, using (i):

$$\begin{aligned}
(\mathbb{E}\,|f(t+\delta) - f(t)|^r)^{\frac{2p\,\alpha}{r}} &\geqslant \mathbb{E}\,|f(t+\delta) - f(t)|^{2p} (\mathbb{E}\,|f(t+\delta) - f(t)|^{2p+2})^{-\frac{(1-\alpha)2p}{2p+2}} \\
&\geqslant (C_{2p}\,\delta^{2H}) (C'_{2p+2}\,\delta^{2H})^{-\frac{(1-\alpha)2p}{2p+2}} \\
&\geqslant C''_r (\delta^{2H})^{1-\frac{(1-\alpha)2p}{2p+2}} = C''_r (\delta^{2H})^{\frac{2p\,\alpha}{r}}
\end{aligned}$$

assuming $0 \leqslant \delta < \min(\delta_{2p}, \delta_{2p+2})$. Hence

$$\mathbb{E}\,|f(t+\delta) - f(t)|^r \geqslant (C''_r)^{\frac{r}{2p\,\alpha}}\,\delta^{2H} = C_r\,\delta^{2H}.$$

Finally for $0 \leqslant \delta < \delta_r = \min(\delta_{2p}, \delta_{2p+2})$ we have $C_r \delta^{2H} \leqslant \mathbb{E}\,|f(t+\delta) - f(t)|^q \leqslant C'_r \delta^{2H}$ as required. □

Let us observe that, contrarily to B_H, the exponent-function $q \mapsto H(q)$ is not linear on $[2, +\infty)$. The function is constant ($H(q) = 2H$) over this interval. We may think about this as a high irregularity statistical indicator for F.

3.2. **Quadratic variations.**

As suggested by (17), the study of increments leads naturally to estimate the Hurst index. In practice, when only discrete observations are available, one can use either wavelets as in [8], or q-variations. In the Gaussian framework, asymptotic properties of the estimators are obtained from the quadratic case $q = 2$ using Hermite expansions of the function $|\cdot|^q$ [11, 9].

For any $t_0 \in \mathbb{R}^D$ and $\theta \in S^{D-1}$ we consider here the discrete 2-structure function of the line process $L_{t_0,\theta}F$ instead of (14), replacing the integral by a finite sum and choosing $\delta = 2^{-n}u$ with u a positive integer. It leads to the quadratic variations of $L_{t_0,\theta}F$ with step $u \in \mathbb{N} \setminus \{0\}$:

$$V_n(u) = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \left(F(t_0 + (2^{-n}(k+u))\theta) - F(t_0 + (2^{-n}k)\theta)\right)^2. \tag{21}$$

In order to compute the asymptotic properties of $V_n(u)$ we introduce the stationary sequence

$$X_n(k) = F(t_0+(2^{-n}(k+u))\theta) - F(t_0+(2^{-n}k)\theta) = \int_{\mathbb{R}^D \times \mathbb{R}^+} \psi(2^{-n}u\theta, x - t_0 - 2^{-n}k\theta, r)\, N(dx, dr)$$

according to (18). Note that $V_n(u)$ is then the empirical mean of $(X_n^2(k))_{0 \leqslant k \leqslant 2^n - 1}$ so that

$$\mathbb{E}(V_n(u)) = \mathbb{E}(X_n^2(k)) = \int_{\mathbb{R}^D \times \mathbb{R}^+} \psi_{n,k}^2(x, r) \nu(dx, dr) = c_H (2^{-n}u)^{2H}. \tag{22}$$

Then, a natural estimator of H can be built by considering log-log ratios of $V_n(\cdot)$, precisely we will prove the following theorem.

Theorem 3.2. *Let $u, v \geqslant 1$ with $u \neq v$. Then, almost surely as $n \to +\infty$,*

$$\widehat{H}_n(u,v) = \frac{1}{2}\log\left(\frac{V_n(u)}{V_n(v)}\right) / \log\left(\frac{u}{v}\right) \longrightarrow H$$

where $V_n(u)$ is defined by (21).

Proof. Using (22), it is enough to prove that almost surely, as $n \to +\infty$,

$$\frac{V_n(u)}{\mathbb{E}(V_n(u))} \to 1.$$

By Markov inequality we have, for all $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\frac{V_n(u)}{\mathbb{E}(V_n(u))} - 1\right| > \varepsilon\right) \leqslant \frac{\mathrm{Var}(V_n(u))}{\varepsilon^2 \mathbb{E}(V_n(u))^2},$$

thus the only thing to do is to control the variance of $V_n(u)$ in such a way that we can use Borel-Cantelli lemma. First let us remark that by stationarity

$$\mathrm{Var}(V_n(u)) = \frac{1}{2^{2n}} \sum_{k,l=0}^{n-1} \mathrm{Cov}\left(X_n^2(k), X_n^2(l)\right) = \frac{1}{2^n} \sum_{k=-(2^n-1)}^{2^n-1} (1 - 2^{-n}|k|)\mathrm{Cov}\left(X_n^2(0), X_n^2(|k|)\right).$$

To compute the covariances we follow the framework of [17]. We can write $X_n(k) = I_1(\psi_{n,k})$ as the Wiener-Itô integral of

$$\psi_{n,k}(x,r) = \psi(2^{-n}u\theta, x - t_0 - 2^{-n}k\theta, r) \tag{23}$$

with respect to the compensated Poisson random measure $N - \nu$ on $\mathbb{R}^D \times \mathbb{R}^+$. Moreover, according to the product formula we have

$$X_n^2(k) = I_2\left(\psi_{n,k} \otimes \psi_{n,k}\right) + I_1\left(\psi_{n,k}^2\right) + \int_{\mathbb{R}^D \times \mathbb{R}^+} \psi_{n,k}^2(x,r)\nu(dx,dr) \tag{24}$$

where $I_2\left(\psi_{n,k} \otimes \psi_{n,k}\right)$ is the multiple Wiener-Itô integral of order 2 of the symmetric function $\psi_{n,k} \otimes \psi_{n,k} \in L^2(\left(\mathbb{R}^D \times \mathbb{R}^+\right)^2, \nu(dx,dr)^{\otimes 2})$. Then, using (24), by isometry we obtain that, for $k > 0$,

$$\begin{aligned}
&\mathrm{Cov}\left(X_n^2(0), X_n^2(k)\right) \\
&\quad = 2\left(\int_{\mathbb{R}^D \times \mathbb{R}^+} \psi_{n,0}(x,r)\psi_{n,k}(x,r)\nu(dx,dr)\right)^2 + \int_{\mathbb{R}^D \times \mathbb{R}^+} \psi_{n,0}^2(x,r)\psi_{n,k}^2(x,r)\nu(dx,dr).
\end{aligned}$$

For the first term, let us remark that by a change of variables one has

$$\begin{aligned}
\int_{\mathbb{R}^D \times \mathbb{R}^+} \psi_{n,0}(x,r)\psi_{n,k}(x,r)\nu(dx,dr) &= \mathrm{Cov}\left(F(2^{-n}u\theta), F(2^{-n}(u+k)\theta) - F(2^{-n}k\theta)\right) \\
&= (2^{-n})^{2H}\rho_u(k)
\end{aligned}$$

with

$$\rho_u(k) = \mathrm{Cov}\left(F(u\theta), F((u+k)\theta) - F(k\theta)\right) = \frac{c_H}{2}\left(|u+k|^{2H} - 2|k|^{2H} + |u-k|^{2H}\right) \tag{25}$$

according to (5). Similarly, by a change of variables the second term satisfies

$$\int_{\mathbb{R}^D \times \mathbb{R}^+} \psi_{n,0}^2(x,r)\psi_{n,k}^2(x,r)\nu(dx,dr) = (2^{-n})^{2H}\widetilde{\rho_u}(k)$$

with

$$\widetilde{\rho_u}(k) = \int_{\mathbb{R}^D\times\mathbb{R}^+} \left(\mathbb{1}_{B(x,r)}(u\theta) - \mathbb{1}_{B(x,r)}(0)\right)^2 \left(\mathbb{1}_{B(x,r)}((u+k)\theta) - \mathbb{1}_{B(x,r)}(k\theta)\right)^2 \nu(dx,dr). \tag{26}$$

Let us write

$$\mathbb{1}_{B(x,r)}(u\theta) - \mathbb{1}_{B(x,r)}(0) = \mathbb{1}_{\mathcal{C}(u\theta)\cap\mathcal{C}(0)^c}(x,r) - \mathbb{1}_{\mathcal{C}(0)\cap\mathcal{C}(u\theta)^c}(x,r)$$

where $\mathcal{C}(u\theta)$ is the cone defined by (6). Note that $\widetilde{\rho_u}$ is even and let us consider $k > u$. We set $\mathring{T} = (0, u\theta, k\theta, (u+k)\theta)$ so that, according to (9), we can write the integrand in (26) as the sum of indicator functions of the following sets: $C(\mathring{T}, (0,1,0,1))$, $C(\mathring{T}, (0,1,1,0))$, $C(\mathring{T}, (1,0,0,1))$ and $C(\mathring{T}, (1,0,1,0))$. Since $k > u$, each of them is empty except $C(\mathring{T}, (0,1,1,0))$ and hence

$$\begin{aligned}\left(\mathbb{1}_{B(x,r)}(u\theta) - \mathbb{1}_{B(x,r)}(0)\right)^2 &\left(\mathbb{1}_{B(x,r)}((u+k)\theta) - \mathbb{1}_{B(x,r)}(k\theta)\right)^2 \\ &= \mathbb{1}_{\mathcal{C}(u\theta)\cap\mathcal{C}(0)^c\cap\mathcal{C}((u+k)\theta)^c\cap\mathcal{C}(k\theta)} \\ &= -\left(\mathbb{1}_{B(x,r)}(u\theta) - \mathbb{1}_{B(x,r)}(0)\right)\left(\mathbb{1}_{B(x,r)}((u+k)\theta) - \mathbb{1}_{B(x,r)}(k\theta)\right).\end{aligned}$$

Therefore, by symmetry, for any $|k| > u$ we have $\widetilde{\rho_u}(k) = -\rho_u(k)$.
Finally

$$\mathrm{Var}(V_n(u)) = (2^{-n})^{4H+1}\, 2 \sum_{k=-(2^n-1)}^{2^n-1} (1-2^{-n}|k|)\rho_u^2(k) + (2^{-n})^{2H+1} \sum_{k=-(2^n-1)}^{2^n-1} (1-2^{-n}|k|)\widetilde{\rho_u}(k).$$

Note that (25) implies that $\rho_u(k) \underset{|k|\to+\infty}{=} O\left(|k|^{-2(1-H)}\right)$ and hence $\rho_u \in l^2(\mathbb{Z})$ and $\widetilde{\rho_u} \in l^1(\mathbb{Z})$. Thus, the first term of $\mathrm{Var}(V_n(u))$ is equivalent to $(2^{-n})^{4H+1}\, 2\sum_{k\in\mathbb{Z}} \rho_u^2(k)$ and the second one is equivalent to $(2^{-n})^{2H+1} \sum_{k\in\mathbb{Z}} \widetilde{\rho_u}(k)$. Using (22), it yields

$$\frac{\mathrm{Var}(V_n(u))}{\mathbb{E}(V_n(u))^2} \underset{n\to+\infty}{\sim} (2^{-n})^{1-2H} \frac{\sum_{k\in\mathbb{Z}} \widetilde{\rho_u}(k)}{c_H^2 u^{4H}}.$$

Since $H < 1/2$, the Borel-Cantelli Lemma allows us to conclude the proof. □

References

[1] Bassan B. and Bona E., *Moments of stochastic processes governed by Poisson random measures*, Commentationes Mathematicae Universitatis Carolinae, **31**, n°2, 337–343, (1990).

[2] Biermé H., Estrade A. and Kaj I., *Self-similar random fields and rescaled random balls model*, Journ. Theor. Prob., **23**, 1110–1141 (2010).

[3] Breton J. C. and Dombry C., *Rescaled weighted random ball models and stable self-similar random fields*, Stochastic Process. Appl., **119**, n°10, 3633–3652, (2009).

[4] Chainais P., *Infinitely Divisible Cascades to Model the Statistics of Natural Images*, IEEE Trans. Pattern analysis and Machine intelligence, **29**, n°12, 2105–2119, (2007).

[5] Cioczek-Georges R. and Mandelbrot B. B., *A class of micropulses and antipersistent fractional Brownian motion*, Stoch. Proc. and Appl. **60**, 1–18, (1995).

[6] Cohen S. and Taqqu M., *Small and large scale behavior of the Poissonized Telecom Process*, Methodology and Computing in Applied Probability **6**, 363–379, (2004).

[7] Duvernet L., *Convergence of the structure function of a multifractal random walk in a mixed asymptotic setting*, Stoch. Anal. and Appl. **28**(5) 763–792, (2010).

[8] Faÿ G., Roueff F. and Soulier P., *Estimation of the memory parameter of the infinite Poisson process*, Bernoulli **13**(2), 473–491, (2007).

[9] Guyon X. and León J. R., *Convergence en loi des H-variations d'un processus gaussien stationnaire sur* $\mathbb{R}$, Ann. Inst. Henri Poincaré, Prob. Stat., **25**(3), 265–282, (1989).
[10] Heinrich L. and Schmidt V., *Normal convergence of multidimensional shot noise and rates of this convergence*, Adv. Appl. Prob., **17**, 709–730, (1985).
[11] Istas J. and Lang G., *Quadratic variations and estimation of the local Holder index of a Gaussian process*, Ann. Inst. Henri Poincaré, Prob. Stat., **33**(4), 407–436, (1997).
[12] Jaffard S., *Multifractal formalism for functions*, SIAM J. Math Anal. **24**, 944–970, (1997).
[13] Kaj I., Leskelä L., Norros I. and Schmidt V., *Scaling limits for random fields with long-range dependence*, Ann. Probab. **35**(2), 528–550, (2007).
[14] Kaj I., Limiting fractal random processes in heavy-tailed systems. In *Proc. Fractals in Engineering V, Tours, France*, 199-217, (2005).
[15] Kaj I. and Taqqu M.S., *Convergence to fractional Brownian motion and to the Telecom process : the integral representation approach*, Prog. Probab. **60**, 383-427, (2008).
[16] Ludeña C., *Confidence intervals for the scaling function of multifractal random walks*, Stat. and Prob. Letters **79**, 1186–1193, (2009).
[17] Peccati G. and Taqqu M. *Central limit theorems for double Poisson integrals*, Bernoulli **14**(3), 791–821, (2008).
[18] Sato Y. *Distributions of stable random fields of Chentsov type*, Nagoya Math. J. **123**, 119–139, (1991).
[19] Samorodnitsky G. and Taqqu M.S., *Stable non-Gaussian random processes*, Chapman & Hall (1994).
[20] Stoyan D., Kendall W. S. and Mecke J., *Stochastic Geometry and its Applications*, J. Wiley, (1989).
[21] Takenaka S., *Integral-geometric construction of self-similar stable processes*, Nagoya Math. J. **123**, 1–12, (1991).
[22] Tricot C., *Function norms and fractal dimension*, SIAM J. Math Anal. **28**, 189–212, (1997).

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