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# Correction to: Nonparametric Laguerre estimation in the multiplicative censoring model

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**Abstract:** The paper "Nonparametric Laguerre estimation in the multiplicative censoring model", *Electronic Journal of Statistics*, 2016, **10**, 3114-3152, contains a wrong statement. We localize the place of the error and give a correct proof.

## 1. Introduction

Observations are either i.i.d. nonnegative data $X_1, \dots, X_n$, with unknown density $f$, or drawn from the model $Y_i = X_i U_i,\ i = 1, \dots, n$ where the $U_i$'s are *i.i.d.* with $\beta(1, k)$ density, $k \geq 1$. The sequences $(X_i), (U_i)$, are independent.

The paper studies projection estimators of $f$ using Laguerre basis and proves upper bounds for the integrated $\mathbb{L}^2$-risk. These upper bounds allow to compute rates on Sobolev-Laguerre balls. Corresponding lower bounds are stated in the case of direct observations and indirect observations with uniform noise.

In Comte and Genon-Catalot (2017), we prove that the upper bounds can be improved, thanks to more precise properties of the Laguerre functions (Askey and Wainger (1965)). They yield better rates on Sobolev-Laguerre balls. In the case of direct observations, these rates turn out to be the classical ones, even though the regularity spaces are not the standard ones. Consequently, the lower bounds proved in the paper (Theorem 3.1 and 3.2) do not hold true.

Section 2 presents the improved upper bounds. For what concerns the lower bounds, we point out the error of the published proof, and give a correct one in Section 3.

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## 2. Improved upper bounds

Consider $X_1, \dots, X_n$ i.i.d. nonnegative random variables with unknown density $f$ belonging to $\mathbb{L}^2(\mathbb{R}^+)$. For each $m \geq 0$, a projection estimator of $f$ is defined by $\hat{f}_m^X = \sum_{j=0}^{m-1} \hat{a}_j \varphi_j$, where $\hat{a}_j^X = \frac{1}{n}\sum_{i=1}^n \varphi_j(X_i) j = 0, \dots, m-1$, and $(\varphi_j)_{j\geq 0}$ is the Laguerre basis defined in Section 2.1 of [2]. The following risk bound is proved in [3].

**Proposition 2.1.** *If $\mathbb{E}(1/\sqrt{X_1}) < +\infty$, we have, for $m$ large enough,*

$$\mathbb{E}(\|\hat{f}_m^X - f\|^2) \leq \|f - f_m\|^2 + C\frac{\sqrt{m}}{n}, \tag{2.1}$$

*for $C$ a constant depending on $\mathbb{E}(1/\sqrt{X_1})$, but not on $m$, where $\|.\|$ is the $\mathbb{L}^2$-norm on $\mathbb{L}^2(\mathbb{R}^+)$.*

Compared with Proposition 2.4 p.3120 of Belomestny *et al.* (2016) with $k = 0$ which is the case of direct observations, the variance term here is upper bounded by $\sqrt{m}/n$ instead of $m/n$. On the other hand, we did not assume $\mathbb{E}(1/\sqrt{X_1}) < +\infty$. Note that the function proposals of Theorem 3.1 in Belomestny *et al.* (2016) satisfy this additional moment assumption.

Now, for $f \in W^s(D)$, the Sobolev Laguerre ball being defined by

$$W^s(D) = \{h : (0, +\infty) \to \mathbb{R}, h \in \mathbb{L}^2((0, +\infty)), |h|_s^2 := \sum_{k\geq 0} k^s a_k^2(h) \leq D\}, \tag{2.2}$$

we have $\|f - f_m\|^2 = \sum_{j\geq m} a_j^2(f) \leq Dm^{-s}$. Therefore, choosing $m_{\text{opt}} = [n^{1/(s+1/2)}]$ in the r.h.s. of (2.1) implies

$$\mathbb{E}(\|\hat{f}_{m_{\text{opt}}}^X - f\|^2) \lesssim n^{-s/(s+1/2)} = n^{-2s/(2s+1)}.$$

This upper bound is thus better than the one obtained in Corollary 3.1 p.3122 for $k = 0$. This is why the lower bound stated in Theorem 3.1 p.3123 is not correct. Note that the new rates can not be improved as they reach the standard rates on classical Sobolev spaces.

Assume now that the observations are $Y_i = X_i U_i$ with $U_i$ i.i.d. with $(X_i)_{1\leq i\leq n}$ and $(U_i)_{1\leq i\leq n}$ independent. We restrict ourselves to $U_1 \sim \mathcal{U}([0, 1])$. We recall that in this case the projection estimator is defined by $\hat{f}_m(x) = \sum_{j=1}^m \hat{a}_j \varphi_j(x)$, with $\hat{a}_j = \frac{1}{n}\sum_{i=1}^n [Y_i \varphi_j'(Y_i) + \varphi_j(Y_i)]$. Then the following risk bound holds (see [3]).

**Proposition 2.2.** *Assume that $\mathbb{E}(X_1) < +\infty$ and $\mathbb{E}(1/\sqrt{X_1}) < +\infty$. For $m$ large enough, we have*

$$\mathbb{E}(\|\hat{f}_m - f\|^2) \leq \|f - f_m\|^2 + c\frac{m^{3/2}}{n}, \tag{2.3}$$

*where $c$ is a constant which depends on $\mathbb{E}(X_1)$ and $\mathbb{E}(1/\sqrt{X_1})$, but not on $m$.*

The rate obtained in Proposition 2.4 p.3120 for the variance term in the case for $k=1$, which is the case of $U_i$ following a uniform distribution, was $m^3/n$, without moment assumptions. Here, the variance term is proved to be of order $m^{3/2}/n$.

Now, for $f \in W^s(D)$ defined by (2.2), choosing $m_{\text{opt}} = [n^{1/(s+3/2)}]$ in the r.h.s. of (2.3) implies

$$\mathbb{E}(\|\hat{f}_{m_{\text{opt}}} - f\|^2) \lesssim n^{-s/(s+3/2)} = n^{-2s/(2s+3)}.$$

This implies that the lower bound stated in Theorem 3.2 p.3124 is not correct.

## 3. Lower bounds

### 3.1. What is wrong?

We made a wrong use of Theorem 2.6 p.100 in Tsybakov (2009). This theorem requires (Condition $(ii)$), that

$$\frac{1}{M}\sum_{j=1}^{M} \chi^2(P_{\boldsymbol{\theta}^{(j)}}, P_{\boldsymbol{\theta}^{(0)}}) \le \alpha M, \quad 0<\alpha<\frac{1}{2},$$

where $P_{\boldsymbol{\theta}^{(j)}}, j=0,1,\ldots,M$ are the law of $(X_1,\ldots,X_n)$ when the $X_i$'s are i.i.d. with density $f_{\boldsymbol{\theta}^{(j)}}$. This means that $P_{\boldsymbol{\theta}^{(j)}} = f_{\boldsymbol{\theta}^{(j)}}^{\otimes n}$. Observe that (see p.86 of Tsybakov (2009)):

$$\chi^2(f_{\boldsymbol{\theta}^{(j)}}^{\otimes n}, f_{\boldsymbol{\theta}^{(0)}}^{\otimes n}) = \big(1+\chi^2(f_{\boldsymbol{\theta}^{(j)}}, f_{\boldsymbol{\theta}^{(0)}})\big)^n - 1.$$

For $x_n \ge 0$, we have $(1+x_n)^n - 1 = \exp(n\log(1+x_n)) - 1 \le e^{nx_n} - 1$. If $x_n \lesssim \log(M)/n$, then $e^{nx_n} - 1 \lesssim M$. Thus if we prove that $\chi^2(f_{\boldsymbol{\theta}^{(j)}}, f_{\boldsymbol{\theta}^{(0)}}) \lesssim \log(M)$, condition $(ii)$ holds and we can apply Theorem 2.6 of Tsybakov (2009).

On the other hand, if $x_n \lesssim \log^a(M)/n$, with $a>1$, $e^{nx_n}$ has order $e^{\log^a(M)} \gg M$, and condition $(ii)$ is not satisfied. In other words, Lemma 6.5 p.3137 and Lemma 6.12 p.3146 do not hold true, as only the univariate $\chi^2$ is studied and bounded by $\log^a(M)$ with $a>1$. The extrapolation we did to the $n$-sample is not correct.

### 3.2. Corrected lower bounds

Thanks to constructive discussions with Cristina Butucea[1] and Céline Duval[2], we found the error and a new proof corresponding to the correct lower bound.

We treat the case $k=0$ (direct observations of $X_i$). The case of indirect observations can be dealt analogously.

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**Theorem 3.1.** *Assume that $s$ is an integer, $s \geq 1$ and that a $n$ sample $(X_1, \dots, X_n)$ is observed. Then for any estimator $\hat{f}_n$ of $f$ based of $X_1, \dots, X_n$, and for $n$ large enough,*

$$\sup_{f \in W^s(D)} \mathbb{E}_f \left[ \|\hat{f}_n - f\|^2 \right] \gtrsim n^{-2s/(2s+1)}.$$

## 4. Proofs of Theorem 3.1

Let $f_0(x)$ be defined by

$$f_0(x) = \frac{1}{2}\mathbf{1}_{[0,1]}(x) + P(x)\mathbf{1}_{]1,2]}(x)$$

where $P$ is a polynomial such that $P(x) \geq 0$, $\int_1^2 P(x)dx = 1/2$, $P(1) = 1/2$, $P(2) = 0$ and $P^{(k)}(1) = P^{(k)}(2) = 0$ for $k = 1, \dots, s+1$.

Next we consider the functions, for $K \in \mathbb{N}$,

$$f_{\boldsymbol{\theta}}(x) = f_0(x) + \delta K^{-\gamma} \sum_{k=0}^{K-1} \theta_{k+1} \psi(xK - k)$$

for some $\delta > 0$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_K) \in \{0,1\}^K$, $\gamma > 0$ to be chosen and $\psi$ is bounded, has support $[0,1]$, admits bounded derivatives up to order $s$ and $\int_0^1 \psi(x)dx = 0$. The moment condition of Proposition 2.1 holds for these densities.

**Lemma 4.1.** *Let $s$ integer, $s \geq 1$. Then $f_0$ and $f_{\boldsymbol{\theta}}$ are densities belonging to $W^s(D)$ provided that $\gamma \geq s$ and $\delta$ well chosen.*

**Proof of Lemma 4.1.** First $f_0$ is a density, and $\int_{\mathbb{R}^+} f_{\boldsymbol{\theta}}(x)dx = \int_{\mathbb{R}^+} f_0(x)dx = 1$ by construction.

We now prove that $f_{\boldsymbol{\theta}}$ is nonnegative. For any $x \in [0,1]$, then there exists $k_0$ such that $x \in [k_0/K, (k_0+1)/K]$ and we have

$$f_{\boldsymbol{\theta}}(x) = \frac{1}{2} + \delta K^{-\gamma} \theta_{k_0+1} \psi(xK - k_0) \geq \frac{1}{2} - \delta \|\psi\|_\infty K^{-\gamma}.$$

Thus $f_{\boldsymbol{\theta}}(x) \geq 0$ as soon as $\gamma > 0$ and $\delta < 1/(2\|\psi\|_\infty)$.

Now we prove that $f_0$ and $f_{\boldsymbol{\theta}}$ belong to $W^s(D)$. The computation of norms in Sobolev-Laguerre spaces are detailed in paragraph 7.2 of Belomestny *et al.* (2016). For $f_0$, we recall that

$$\|f_0\|_s^2 = \int_0^{+\infty} \left( x^{s/2} \sum_{j=0}^{s} \binom{s}{j} f_0^{(j)}(x) \right)^2 dx \leq 2^s \sum_{j=0}^{s} \binom{s}{j} \int_0^2 \left( x^{s/2} f_0^{(j)}(x) \right)^2 dx.$$

for $j = 0, \dots, s$ and there exists a constant $B(s)$ such that $\|f_0\|_s^2 \leq B(s)$. It follows Lemma 7.5 of Belomestny *et al.* that $|f_0|_s^2 \leq \tilde{B}(s)$, for a constant $\tilde{B}(s)$. We take $D/4 \geq \tilde{B}(s)$.

$$\begin{aligned}\|f_{\boldsymbol{\theta}}-f_0\|_s^2 &= \delta^2 K^{-2\gamma}\int_0^{+\infty}\left(x^{s/2}\sum_{j=0}^{s}\binom{s}{j}\sum_{k=0}^{K-1}\theta_{k+1}K^j\psi^{(j)}(xK-k)\right)^2 dx\\ &\leq \delta^2 K^{-2\gamma}2^s\sum_{j=0}^{s}\binom{s}{j}\int_0^{+\infty}\left(x^{s/2}\sum_{k=0}^{K-1}\theta_{k+1}K^j\psi^{(j)}(xK-k)\right)^2 dx.\end{aligned}$$

We now use that $\psi^{(j)}(xK-k),\psi^{(j)}(xK-\ell)$ have disjoint supports and are bounded (say by $c$). We get

$$\begin{aligned}\|f_{\boldsymbol{\theta}}-f_0\|_s^2 &\leq \delta^2 2^s K^{-2\gamma}c^2\sum_{j=0}^{s}\binom{s}{j}\sum_{k=0}^{K-1}K^{2j}\int_{k/K}^{(k+1)/K}x^s dx\\ &\leq \delta^2 2^{2s}K^{-2\gamma}c'\sum_{j=0}^{s}\sum_{k=0}^{K-1}K^{2j-1}\leq C(s)\delta^2K^{-2\gamma+2s}.\end{aligned}$$

Thus, for $\delta$ small enough, $|f_{\boldsymbol{\theta}}-f_0|_s^2\leq D/4$ as $\gamma\geq s$. □

Next we have:

**Lemma 4.2.** *For any* $\boldsymbol{\theta},\boldsymbol{\theta}'\in\{0,1\}^K$,

$$\int_0^\infty \left(f_{\boldsymbol{\theta}}(x)-f_{\boldsymbol{\theta}'}(x)\right)^2 dx \geq \delta^2\|\psi\|^2(2K)^{-2\gamma-1}\rho(\boldsymbol{\theta},\boldsymbol{\theta}'), \tag{4.1}$$

*where* $\rho(\boldsymbol{\theta},\boldsymbol{\theta}')=\sum_{k=1}^K \mathbf{1}_{\theta_k\neq\theta'_k}$ *is the so-called Hamming distance.*

**Proof of Lemma 4.2.**

$$\begin{aligned}\|f_{\boldsymbol{\theta}}-f_{\boldsymbol{\theta}'}\|^2 &= \delta^2\int_0^{+\infty}\left(\sum_{k=0}^{K-1}(\theta_{k+1}-\theta'_{k+1})K^{-\gamma}\psi(xK-k)\right)^2 dx\\ &= \delta^2\sum_{k=1}^{K}(\theta_k-\theta'_k)^2K^{-2\gamma-1}\|\psi\|^2=\delta^2\|\psi\|^2K^{-2\gamma-1}\rho(\boldsymbol{\theta},\boldsymbol{\theta}'). \quad \square\end{aligned}$$

We recall the Varshamov-Gilbert bound (see Lemma 2.9 p. 104 in [4]).

**Lemma 4.3.** *Fix some even integer* $K>0$. *There exists a subset* $\{\boldsymbol{\theta}^{(0)},\ldots,\boldsymbol{\theta}^{(M)}\}$ *of* $\{0,1\}^K$ *and a constant* $A_1>0$, *such that* $\boldsymbol{\theta}^{(0)}=(0,\ldots,0)$, $\rho(\theta^{(j)},\theta^{(l)})\geq A_1K$, *for all* $0\leq j<l\leq M$. *Moreover it holds that, for some constant* $A_2>0$,

$$M\geq 2^{A_2K}. \tag{4.2}$$

Therefore

$$\|f_{\boldsymbol{\theta}^{(j)}}-f_{\boldsymbol{\theta}^{(l)}}\|^2\geq C\delta^2K^{-2\gamma}.$$

Then we have the following Lemma.

**Lemma 4.4.** *For* $j \in \{1, \dots, M\}$, $\chi^2\left(f_{\boldsymbol{\theta}^{(j)}}, f_{\boldsymbol{\theta}^{(0)}}\right) \lesssim \delta^2 \log(M) K^{-2\gamma-1}$*, where* $M$ *comes from the Varshamov-Gilbert Lemma.*

**Proof of Lemma 4.4.** We have $f_0 = f_{\boldsymbol{\theta}^{(0)}}$, and

$$\begin{aligned}\chi^2(f_{\boldsymbol{\theta}}, f_0) &= \int_0^1 \frac{\left(f_{\boldsymbol{\theta}}(x) - f_0(x)\right)^2}{f_0(x)}\, dx \\ &\lesssim \delta^2 \sum_{k=0}^{K-1} \theta_{k+1}^2 K^{-2\gamma} \int_0^1 \psi^2(xK - k) dx \lesssim \delta^2 K^{-2\gamma} \|\psi\|^2.\end{aligned}$$

Thus

$$\chi^2(f_{\boldsymbol{\theta}^{(j)}}, f_{\boldsymbol{\theta}^{(0)}}) \lesssim \delta^2 \log(M) K^{-2\gamma-1}. \quad \square$$

So if $\delta^2$ is a well chosen constant, $\gamma = s$ and $K = n^{1/(2\gamma+1)} = n^{1/(2s+1)}$ we get

$$\frac{1}{M} \sum_{j=1}^{M} \chi^2\left((f_{\boldsymbol{\theta}^{(j)}})^{\otimes n}, (f_{\boldsymbol{\theta}^{(0)}})^{\otimes n}\right) \leq \alpha M,$$

for $0 < \alpha < 1/8$, and

$$\|f_{\boldsymbol{\theta}^{(j)}} - f_{\boldsymbol{\theta}^{(l)}}\|^2 \geq C\delta^2 K^{-2\gamma} \propto n^{-2s/(2s+1)}.$$

Applying Theorem 2.6 of Tsybakov (2009) gives the result of Theorem 3.1. □

## References


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