

NONPARAMETRIC DRIFT ESTIMATION FROM DIFFUSIONS WITH CORRELATED BROWNIAN MOTIONS

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ABSTRACT. In the present paper, we consider that N diffusion processes X^1, \dots, X^N are observed on $[0, T]$, where T is fixed and N grows to infinity. Contrary to most of the recent works, we no longer assume that the processes are independent. The dependency is modeled through correlations between the Brownian motions driving the diffusion processes. A nonparametric estimator of the drift function, which does not use the knowledge of the correlation matrix, is proposed and studied. Its integrated mean squared risk is bounded and an adaptive procedure is proposed. Few theoretical tools to handle this kind of dependency are available, and this makes our results new. Numerical experiments show that the procedure works in practice.

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1. INTRODUCTION

We start by describing our model. Consider the diffusion process $X = (X_t)_{t \in [0, T]}$, defined by

$$(1) \quad X_t = x_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s ; t \in [0, T],$$

where $x_0 \in \mathbb{R}$, $B = (B_t)_{t \in [0, T]}$ is a Brownian motion, $b : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz continuous function, and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded Lipschitz continuous function. Now, let B^1, \dots, B^N be $N \in \mathbb{N} \setminus \{0\}$ copies of B such that

$$(2) \quad \mathbb{E}(B_s^i B_t^k) = R_{i,k}(s \wedge t) ; \forall i, k \in \{1, \dots, N\}, \forall s, t \in [0, T],$$

where $R = (R_{i,k})_{i,k}$ is a correlation matrix. Note that, thanks to the (stochastic) integration by parts formula, the dependence condition (2) on B^1, \dots, B^N implies that, for every $i, k \in \{1, \dots, N\}$, $d\langle B^i, B^k \rangle_t = R_{i,k} dt$, with $R_{i,i} = 1$. Finally, consider $X^i := \mathcal{I}(x_0, B^i)$ for every $i \in \{1, \dots, N\}$, where $\mathcal{I}(\cdot)$ is the Itô map associated to Equation (1). In the present paper, we consider that these N diffusion

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processes X^1, \dots, X^N are observed on $[0, T]$, where T is fixed and N grows to infinity, and our aim is to estimate nonparametrically the drift function $b(\cdot)$.

In the case of independent Brownian motions, that is $R = \mathbf{I}_N$ (the $N \times N$ identity matrix), projection least squares estimator have been studied in Comte and Genon-Catalot [7] for continuous time observations, in Denis et al. [14] for discrete time (with small step) observations with a classification purpose in the parametric setting, and in Denis *et al.* [15] in the nonparametric context, for instance. Marie and Rosier [19] propose a kernel based Nadaraya-Watson estimator of the drift function b , with bandwidth selection relying on the Penalization Comparison to Overfitting (PCO) criterion recently introduced by Lacour *et al.* [18]. Still in the case $R = \mathbf{I}_N$, Comte and Marie [12] investigate the properties of the projection least squares estimator of the drift when B is a fractional Brownian motion.

Dependency is often encountered in recent works in the context of stochastic systems of N interacting particles, with recent nonparametric drift estimators proposals in Della Maestra and Hoffmann [13], Belomestny *et al.* [3] or Comte and Genon-Catalot [9]. These kinds of models are related to physics. We rather have in mind economic or financial models. For instance, in Duellmann *et al.* [16], the authors consider a portfolio of N homogenous firms such that the asset value X_t^i at time t of the i -th firm is modeled by Merton's model (see [20]) $dX_t^i = \mu X_t^i dt + \sigma X_t^i dB_t^i$ with $X_0^i = X_0$, which corresponds to (1) with $b(x) = \mu x$ and $\sigma(x) = \sigma x$. Intending to capture the dependency between the firms, they also assume that $dB_t^i = \sqrt{\rho} dW_t + \sqrt{1-\rho} dW_t^i$, where W is a *common systematic risk factor*, W^i is a *firm-specific risk factor* and $\rho \in [0, 1]$. This corresponds to a particular matrix R , precisely $R_{i,k} = \rho$ for $i \neq k$ (and $R_{i,i} = 1$), so that one single parameter ρ represents the so-called *asset correlation*. This model has been considered in e.g. Bush *et al.* [4], for the more mathematical purpose of studying the limit of the empirical distribution of the X_t^i 's (see also references therein). Our extension from specific geometric Brownian motion to general nonparametric diffusion (1), and from one single correlation parameter to a general matrix representation, is therefore standard in both respects. This context has nevertheless never been considered before up to our knowledge. Let us emphasize that our aim is not to estimate R , but to exhibit conditions on it such that $b(\cdot)$ can be estimated with performance near of the independent setting.

In our framework, T is fixed, and N is large. Our results are nonasymptotic, but the idea is that N grows to infinity. We fix a subset I of \mathbb{R} and build a collection of projection least squares estimators of $b_I = b\mathbf{1}_I$ where I is compact or not. The estimators are defined by their coefficients on an orthonormal basis of $\mathbb{L}^2(I)$, $\varphi_1, \dots, \varphi_m$, resulting from a standard least squares computation. Precisely, we consider the estimator of the drift function b minimizing the objective function $\gamma_N(\tau)$

$$(3) \quad \tau \mapsto \gamma_N(\tau) := \frac{1}{NT} \sum_{i=1}^N \left(\int_0^T \tau(X_s^i)^2 ds - 2 \int_0^T \tau(X_s^i) dX_s^i \right)$$

on the m -dimensional function space $\mathcal{S}_m = \text{span}\{\varphi_1, \dots, \varphi_m\}$. Note that the first part of $\gamma_N(\tau)$ involves a quantity

$$\|\tau\|_N^2 := \frac{1}{NT} \sum_{i=1}^N \int_0^T \tau(X_s^i)^2 ds,$$

which is considered as *the squared empirical norm* of the function τ . These estimators are the same as in Comte and Genon-Catalot [7], but their study is made significantly more difficult by the dependency context. We do not have at our disposal any coupling method nor any transformation leading to a simpler system; in particular, applying $R^{-1/2}$ to the system does not bring any simplification because of a "widespreading" of the components of the process. Tropp's deviation inequalities used in the independent context (see Tropp [22], Matrix Chernov Inequality, Theorem 1.1 and Matrix Bernstein Inequality, Theorem 1.4), which allow to consider the empirical norm and its expectation (an integral norm, thus) as equivalent with high probability, no longer apply. Martingale properties still are useful, and we turn to Azuma's matrix deviation inequality (see Tropp [22], Theorem 7.1), which however requires to set sparsity conditions on R (see Assumption 3.4). This equivalence property is the key of the rigorous study of the risk of the drift estimator, and the correlation matrix is therefore at the heart of the proofs.

The plan of the paper is the following. A first parametric example motivates the model and the way of estimating a drift parameter in Section 2. The general nonparametric drift estimator is defined in Section 3 and a risk bound on a fixed projection space is proved. Adaptive estimation is studied in Section 4 and the whole procedure is illustrated through simulations in Section 5. Lastly, proofs are gathered in Section 6.

2. PRELIMINARY MOTIVATION AND EXAMPLE IN THE PARAMETRIC FRAMEWORK

This preliminary section deals with the geometric model described in the introduction, in the parametric framework, in order to motivate our investigations. Similarly to Duellmann *et al.* [16], consider N risky assets of same nature and of prices processes X^1, \dots, X^N observed on the time interval $[0, T]$. Since these assets are of same nature, to model their prices by $dX_t^i = \mu X_t^i dt + \sigma X_t^i dB_t^i$ with $\mu \in \mathbb{R}$ and $\sigma > 0$ not depending on $i \in \{1, \dots, N\}$ is realistic, but it is also very realistic to consider that B^1, \dots, B^N may be dependent, through the correlation matrix described above. Let us compute the quadratic risk of the least squares estimator $\hat{\theta}_N$ of $\theta = \mu - \sigma^2/2$ in this special case. Since we can write that $X_t^i = x_0 \exp(Y_t^i)$ with $Y_t^i = \theta t + \sigma B_t^i$ for every $i \in \{1, \dots, N\}$ and $t \in [0, T]$, we set

$$\hat{\theta}_N = \frac{1}{NT} \sum_{i=1}^N Y_T^i = \theta + \frac{\sigma}{NT} \sum_{i=1}^N B_T^i.$$

Then,

$$\begin{aligned} \mathbb{E}(|\hat{\theta}_N - \theta|^2) &= \frac{\sigma^2}{N^2 T^2} \left[\sum_{i=1}^N \mathbb{E}(|B_T^i|^2) + \sum_{i \neq k} \mathbb{E}(B_T^i B_T^k) \right] \\ &= \frac{\sigma^2}{NT} + \frac{\sigma^2}{N^2 T} \sum_{i \neq k} R_{i,k} = \frac{\sigma^2}{NT} \left(1 + \frac{1}{N} \sum_{i \neq k} R_{i,k} \right). \end{aligned}$$

This means that the rate of convergence of $\hat{\theta}_N$ is of order

$$\mathbf{V} := \frac{1}{N} + \frac{1}{N^2} \sum_{i \neq k} R_{i,k}.$$

We note that if $R_{i,k} = \rho$ for all $i \neq k$, then the estimator is not consistent. This would be the same if all the coefficients of R were positive and only bounded by a constant $\rho > 0$. However, if we set a sparsity condition by saying that R is block-diagonal with blocks of size (less than) k_0 , and if we assume that all nonzero coefficients are equal to (or bounded by) ρ , then $\sum_{i \neq k} R_{i,k} \leq k_0 \rho N$. So, $k_0 \rho$ is the loss in risk due to dependency, while the rate remains $O(1/N)$. Referring to the firms model of Duellmann *et al.* [16] and Bush *et al.* [4], this means for instance that for a large N , dependent firms have to be grouped as several independent sets aggregated in the global model. Another way to model the dependency with few parameters is to assume that $dB_t^i = \sqrt{\mathbf{a}} dW_t^i + \sqrt{1 - \mathbf{a}} dW_t^{i+1}$, where W^1, \dots, W^{N+1} are independent Brownian motions and $\mathbf{a} \in [0, 1]$. This is a way of saying that each firm is correlated to the following one in the list. In that case,

$$R_{i,i+1} = R_{i,i-1} = \sqrt{\mathbf{a}(1 - \mathbf{a})}, \quad R_{i,i} = 1, \quad R_{i,k} = 0 \text{ for } |k - i| > 1;$$

and then

$$\mathbf{V} = \frac{1}{N} \left(1 + 2 \left(1 - \frac{1}{N} \right) \sqrt{\mathbf{a}(1 - \mathbf{a})} \right)$$

has order $O(1/N)$. Note that this matrix is sparse in the sense of Assumption 3.4 below.

Our purpose is to show that, at least for some special dependence schemes on B^1, \dots, B^N , the variance term of the projection (nonparametric) least squares estimator of $b(\cdot)$ introduced in the following section

is at most of order

$$\frac{1}{N} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right).$$

It is noteworthy that the estimator $\hat{\theta}_N$ is the maximum likelihood estimator (MLE) when X^1, \dots, X^N are independent, while the MLE in our dependent setting would involve – and thus require the complete knowledge of – the matrix R (more specifically, its inverse). In the present strategy, the knowledge of R is not required, which is interesting and may justify a loss of efficiency.

3. A PROJECTION LEAST SQUARES ESTIMATOR OF THE DRIFT FUNCTION

3.1. The objective function. Let f_T be the density function defined by

$$f_T(x) := \frac{1}{T} \int_0^T p_s(x_0, x) ds ; \forall x \in \mathbb{R},$$

where $p_s(x_0, \cdot)$ is the density with respect to Lebesgue's measure of the probability distribution of X_s for every $s \in (0, T]$. Let us consider the projection space $\mathcal{S}_m := \text{span}\{\varphi_1, \dots, \varphi_m\}$, where $\varphi_1, \dots, \varphi_{N_T}$ are continuous functions from I into \mathbb{R} such that $(\varphi_1, \dots, \varphi_{N_T})$ is an orthonormal family in $\mathbb{L}^2(I, dx)$, and $I \subset \mathbb{R}$ is a non-empty interval. We recall now that the objective function $\tau \in \mathcal{S}_m \mapsto \gamma_N(\tau)$ is defined by (3), where $m \in \{1, \dots, N_T\}$ and $N_T := [NT] + 1$. We choose a contrast which is the same as in the independent case. Note that for the nonparametric estimation of the drift function from N observed paths, even in the independent case, least squares and maximum likelihood strategies do not match. Indeed, the likelihood would involve weights $\sigma(X_s^i)^{-2}$ inside all integrals. In the dependent case, there would also be the matrix R^{-1} to take into account. Even if both $\sigma(\cdot)$ and R can be considered as known, it is interesting not to need them to compute the drift estimator. In particular, the step to discrete time high frequency data is then much simpler. Since the strategy works in the independent case, we can hope that if the correlations are not too strong, then the strategy remains relevant.

Remark. For any $\tau \in \mathcal{S}_m$,

$$\begin{aligned} \mathbb{E}(\gamma_N(\tau)) &= \frac{1}{T} \int_0^T \mathbb{E}(|\tau(X_s) - b(X_s)|^2) ds - \frac{1}{T} \int_0^T \mathbb{E}(b(X_s)^2) ds \\ &= \int_{-\infty}^{\infty} (\tau(x) - b(x))^2 f_T(x) dx - \int_{-\infty}^{\infty} b(x)^2 f_T(x) dx. \end{aligned}$$

Then, the more τ is close to b , the more $\mathbb{E}(\gamma_N(\tau))$ is small. For this reason, the estimator of b minimizing $\gamma_N(\cdot)$ is studied in this paper.

3.2. The projection least squares estimator and some related matrices. In this section, m is a fixed integer in $\{1, \dots, N_T\}$. We consider the estimator

$$(4) \quad \hat{b}_m := \arg \min_{\tau \in \mathcal{S}_m} \gamma_N(\tau)$$

of b , if it exists and is unique. Since $\mathcal{S}_m = \text{span}\{\varphi_1, \dots, \varphi_m\}$, there exist m square integrable random variables $\hat{\theta}_1, \dots, \hat{\theta}_m$ such that

$$\hat{b}_m = \sum_{j=1}^m \hat{\theta}_j \varphi_j.$$

Then,

$$\nabla \gamma_N(\hat{b}_m) = \left(\frac{1}{NT} \sum_{i=1}^N \left(2 \sum_{\ell=1}^m \hat{\theta}_\ell \int_0^T \varphi_j(X_s^i) \varphi_\ell(X_s^i) ds - 2 \int_0^T \varphi_j(X_s^i) dX_s^i \right) \right)_{j \in \{1, \dots, m\}}.$$

Let

$$\hat{\Psi}_m := \left(\frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \varphi_\ell(X_s^i) ds \right)_{j, \ell \in \{1, \dots, m\}}$$

and

$$\widehat{\mathbf{X}}_m := \left(\frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) dX_s^i \right)_{j \in \{1, \dots, m\}}.$$

Therefore, by (4) and if $\widehat{\Psi}_m$ is invertible, necessarily

$$\widehat{\Theta}_m := (\widehat{\theta}_1, \dots, \widehat{\theta}_m)^* = \widehat{\Psi}_m^{-1} \widehat{\mathbf{X}}_m,$$

where \mathbf{M}^* denotes the transpose of the matrix \mathbf{M} .

Remarks:

- (1) We can write $\widehat{\Psi}_m = (\langle \varphi_j, \varphi_\ell \rangle_N)_{j, \ell}$, where

$$\langle \varphi, \psi \rangle_N := \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi(X_s^i) \psi(X_s^i) ds$$

for every measurable functions φ and ψ from \mathbb{R} into itself.

- (2) The following useful decomposition holds: $\widehat{\mathbf{X}}_m = (\langle b, \varphi_j \rangle_N)_j^* + \widehat{\mathbf{E}}_m$, where

$$\widehat{\mathbf{E}}_m := \left(\frac{1}{NT} \sum_{i=1}^N \int_0^T \sigma(X_s^i) \varphi_j(X_s^i) dB_s^i \right)_{j \in \{1, \dots, m\}}^*.$$

Let us introduce the two following deterministic matrices related to the previous random ones:

- $\Psi_m := \mathbb{E}(\widehat{\Psi}_m) = (\langle \varphi_j, \varphi_\ell \rangle_{f_T})_{j, \ell}$, where $\langle \cdot, \cdot \rangle_{f_T}$ is the scalar product in $\mathbb{L}^2(I, f_T(x) dx)$.
- $\Psi_{m, \sigma} := NT \mathbb{E}(\widehat{\mathbf{E}}_m \widehat{\mathbf{E}}_m^*)$.

Note that under the following assumption, Comte and Genon-Catalot established in [7] (see Lemma 1) that Ψ_m is invertible.

Assumption 3.1. *The φ_j 's satisfy the three following conditions:*

- (1) $(\varphi_1, \dots, \varphi_m)$ is an orthonormal family of $\mathbb{L}^2(I, dx)$.
- (2) The φ_j 's are bounded, continuously derivable, and of bounded derivatives.
- (3) There exist $x_1, \dots, x_m \in I$ such that $\det[(\varphi_j(x_\ell))_{j, \ell}] \neq 0$.

Let us conclude this section with the following suitable bound on the trace of $\Psi_m^{-1/2} \Psi_{m, \sigma} \Psi_m^{-1/2}$. To that aim, we define the following quantity associated with the basis:

$$L(m) := 1 \vee \left(\sup_{x \in I} \sum_{j=1}^m \varphi_j(x)^2 \right).$$

Lemma 3.2. *Under Assumption 3.1, for σ belonging to $\mathbb{L}^2(\mathbb{R}, f_T(x) dx)$ but possibly unbounded,*

$$(5) \quad \text{trace}(\Psi_m^{-1/2} \Psi_{m, \sigma} \Psi_m^{-1/2}) \leq \mathfrak{c}_{3.2} L(m) \|\Psi_m^{-1}\|_{\text{op}} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i, k}| \right)$$

with

$$\mathfrak{c}_{3.2} = \int_{-\infty}^{\infty} \sigma(x)^2 f_T(x) dx.$$

If in addition σ is bounded, then

$$(6) \quad \text{trace}(\Psi_m^{-1/2} \Psi_{m, \sigma} \Psi_m^{-1/2}) \leq m \|\sigma\|_{\infty}^2 \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i, k}| \right).$$

The two previous bounds on the trace compete. In some contexts, they can have the same order m . This occurs if both following conditions hold (this setting is referred to as "compact setting" below):

- I is a compact set and $L(m) = m$, as in the case of a trigonometric basis.

- f_T is lower bounded on I by $f_0 > 0$. Indeed, then $\|\Psi_m^{-1}\|_{\text{op}} \leq 1/f_0$ (see [7]).

However, the bound (5) is not relevant for a non compactly supported basis. For instance, for the Hermite basis described below, $L(m)$ is of order \sqrt{m} , but $\|\Psi_m^{-1}\|_{\text{op}}$ is increasing with m and can be checked to be numerically very large. So, for the Hermite basis, the second bound (6) must be preferred if σ is bounded, and is used in the sequel.

Finally, note that $N^{-1} \sum_{i,k} |R_{i,k}|$ can be replaced by $\|R\|_{\text{op}}$ in the bounds (5) and (6), and it holds that

$$\frac{1}{N} \left| \sum_{i,k=1}^N R_{i,k} \right| \leq \|R\|_{\text{op}}.$$

Therefore, if the coefficients of the matrix R are nonnegative, as in the examples of Section 2 or in the simulation Section 5, then $N^{-1} \sum_{i,k} |R_{i,k}|$ is better than $\|R\|_{\text{op}}$.

3.3. Risk bound on the projection least squares estimator. This section deals, for a given model m , with a risk bound on the truncated estimator

$$\tilde{b}_m := \hat{b}_m \mathbf{1}_{\Lambda_m},$$

where

$$\Lambda_m := \left\{ L(m) (\|\widehat{\Psi}_m^{-1}\|_{\text{op}} \vee 1) \leq \mathfrak{c}_T(p) \frac{NT}{\log(NT)} \right\}$$

with

$$\mathfrak{c}_T(p) = \frac{1}{512T(1+p/2)}, p \geq 12.$$

On the event Λ_m , $\widehat{\Psi}_m$ is invertible because

$$\inf\{\text{sp}(\widehat{\Psi}_m)\} \geq \frac{L(m)}{\mathfrak{c}_T(p)} \cdot \frac{\log(NT)}{NT} > 0,$$

and then \tilde{b}_m is well-defined. Consider

$$\mathcal{I}_N := \{i \in \{2, \dots, N\} : \exists k \in \{1, \dots, i-1\} \text{ such that } R_{i,k} \neq 0\}.$$

In the sequel, m fulfills the following assumption, related to the stability condition introduced in Cohen *et al.* [5] and also used in Comte and Genon-Catalot [7]. Due to dependency, it has to be reinforced by undesirable squares.

Assumption 3.3. $[L(m) (\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^2 \leq \frac{\mathfrak{c}_T(p)}{2} \cdot \frac{NT}{\log(NT)}$.

Note that in the so-called *compact setting* defined above, this condition reduces to

$$m \lesssim \sqrt{\frac{NT}{\log(NT)}},$$

which is similar to the constraint obtained in Baraud [1] (see the condition $N_n \leq K^{-1} \sqrt{n/\log(n)^3}$ in his Theorem 1.1). However, this last condition can be improved in the independent case.

Moreover, a sparsity condition has to be set on $|\mathcal{I}_N|$, and this is again in order to handle dependency.

Assumption 3.4. *There exists a deterministic constant $\mathfrak{c}_{3.4} > 0$, not depending on m and N , such that*

$$|\mathcal{I}_N| \leq \mathfrak{c}_{3.4} N^{\frac{1}{2} - \frac{6}{p}}.$$

Remarks:

- (1) The dependence condition (2) on B^1, \dots, B^N ensures that

$$\mathcal{I}_N = \{i \in \{1, \dots, N\} : B^i \text{ is not independent of } \mathcal{F}_{i-1}\},$$

where $\mathcal{F}_0 := \mathcal{F}$ and $\mathcal{F}_i := \sigma(B^1, \dots, B^i)$ for every $i \in \{1, \dots, N\}$. This is crucial in the proof of Theorem 3.5.

(2) The condition on $|\mathcal{I}_N|$ in Assumption 3.4 can be understood as a sparsity type condition on the correlation matrix R . Note that under Assumption 3.4, for N large enough,

$$(7) \quad \frac{|\mathcal{I}_N|^p}{N^{p/2} \log(NT)^{p/2}} \leq \frac{\mathfrak{c}_{3.4}(p)}{N^6} \quad \text{with} \quad \mathfrak{c}_{3.4}(p) = \mathfrak{c}_{3.4}^p.$$

Note that p is also involved in Assumption 3.3 through $\mathfrak{c}_T(p)$. Therefore, the larger p , the larger \mathcal{I}_N , but the smaller the *authorized* choices of m . For Theorem 3.5, the constraint $p \geq 12$ may be lightened into $p \geq 4$ and Assumption 3.4 into

$$|\mathcal{I}_N| \leq \mathfrak{c}_{3.4} N^{\frac{1}{2} - \frac{2}{p}}.$$

However, Theorem 4.3 is more demanding.

Theorem 3.5. *Under Assumptions 3.1, 3.3 and 3.4, there exists a deterministic constant $\mathfrak{c}_{3.5} > 0$, not depending on m and N , such that*

$$\mathbb{E}(\|\tilde{b}_m - b_I\|_N^2) \leq \min_{\tau \in \mathcal{S}_m} \|\tau - b_I\|_{f_T}^2 + \frac{\mathfrak{c}_{3.5} m}{NT} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right) + \frac{\mathfrak{c}_{3.5}}{N}.$$

As usual for a nonparametric estimator, the risk bound involves a bias term

$$\min_{\tau \in \mathcal{S}_m} \|\tau - b_I\|_{f_T}^2,$$

and a variance term of order $m/(NT)$ if

$$\frac{1}{N} \sum_{i,k=1}^N |R_{i,k}|$$

is bounded by a constant which does not depend on N . The last term is of order $1/N$ and gathers all negligible quantities. The larger m , the better the approximation of b_I in \mathcal{S}_m and the smaller the bias. On the opposite, the variance increases with m . This is why a compromise must be done, either theoretically as in Section 2.4 of Comte and Genon-Catalot [7] from which consistency follows, or by a model selection procedure, as described hereafter.

4. MODEL SELECTION

Throughout this section, $(\varphi_1, \dots, \varphi_{N_T})$ and the $R_{i,k}$'s fulfill the following additional assumptions.

Assumption 4.1. *The φ_j 's satisfy the two following (additional) condition:*

(1) *There exists a deterministic constant $\mathfrak{c}_\varphi \geq 1$, not depending on N , such that for every $m \in \{1, \dots, N_T\}$,*

$$L(m) = 1 \vee \left(\sup_{x \in I} \sum_{j=1}^m \varphi_j(x)^2 \right) \leq \mathfrak{c}_\varphi^2 m.$$

(2) *For every $m, m' \in \{1, \dots, N_T\}$, if $m > m'$, then $\mathcal{S}_{m'} \subset \mathcal{S}_m$.*

Assumption 4.2. *There exists a deterministic constant $\mathfrak{m}_{4.2} > 0$, not depending on N , such that*

$$\|R\|_{\text{op}} \leq \mathfrak{m}_{4.2}.$$

Let us consider

$$\hat{m} = \arg \min_{m \in \mathcal{M}_N} \{-\|\hat{b}_m\|_N^2 + \text{pen}(m)\},$$

where

$$\text{pen}(m) := \mathfrak{c}_{\text{cal}} \frac{m}{NT} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right); \quad \forall m \in \{1, \dots, N_T\},$$

$\mathfrak{c}_{\text{cal}} > 0$ is a deterministic constant to calibrate,

$$\widehat{\mathcal{M}}_N := \left\{ m \in \{1, \dots, N_T\} : [\mathfrak{c}_\varphi^2 m (\|\widehat{\Psi}_m^{-1}\|_{\text{op}} \vee 1)]^2 \leq \mathfrak{d}_T(p) \frac{NT}{\log(NT)} \right\}$$

and

$$\mathfrak{d}_T(p) := \frac{1}{512c_\varphi^4 T(1+p/2)}.$$

Consider also the theoretical counterpart

$$\mathcal{M}_N := \left\{ m \in \{1, \dots, NT\} : [c_\varphi^2 m (\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^2 \leq \frac{\mathfrak{d}_T(p)}{4} \cdot \frac{NT}{\log(NT)} \right\} \text{ of } \widehat{\mathcal{M}}_N.$$

Theorem 4.3. *Under Assumptions 3.1, 3.4, 4.1 and 4.2, there exists a deterministic constant $c_{4.3} > 0$, not depending on N , such that*

$$\mathbb{E}(\|\widehat{b}_m - b_I\|_N^2) \leq c_{4.3} \min_{m \in \mathcal{M}_N} \left\{ \min_{\tau \in \mathcal{S}_m} \|\tau - b_I\|_{f_\tau}^2 + \frac{m}{NT} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right) \right\} + \frac{c_{4.3}}{N}.$$

It follows from Theorem 4.3 that the adaptive estimator \widehat{b}_m automatically reaches a squared bias-variance compromise on the collection \mathcal{M}_N .

5. NUMERICAL EXPERIMENTS

In this section, we study the influence of dependency on the performance of the adaptive estimator. We consider two bases:

- The cosine basis on $I = [a, b]$, defined by $\varphi_1(x) := (b-a)^{-1/2} \mathbf{1}_{[a,b]}(x)$, $\varphi_j(x) := (2/(b-a))^{1/2} \cos(\pi j(x-a)/(b-a)) \mathbf{1}_{[a,b]}(x)$ for $j \geq 2$. The interval $[a, b]$ is chosen different for each model. The basis is orthonormal and fulfills $\sum_{j=1}^m \varphi_j^2(x) \leq 2m$.
- The Hermite basis on $I = \mathbb{R}$, defined from the Hermite polynomials H_j and given by

$$H_j(x) := (-1)^j e^{x^2} \frac{d^j}{dx^j} (e^{-x^2}), \quad \varphi_j(x) := c_{j-1} H_{j-1}(x) e^{-x^2/2}, \quad c_j = (2^j j! \sqrt{\pi})^{-1/2}.$$

The sequence $(\varphi_j)_{j \geq 0}$ is an orthonormal bounded basis of $\mathbb{L}^2(\mathbb{R}, dx)$ with $|\varphi_j(x)| \leq 1/\pi^{1/4}$ (see Indritz [17]). It is proved in Comte and Lacour [11] (see Lemma 1) that $L(m) \leq K\sqrt{m}$ for some constant K .

We experiment five models, where I is the chosen domain of representation for the Hermite basis, and the basis support for the cosine basis:

- (1) Hyperbolic diffusion, $b_1(x) = -\theta x$ and $\sigma_1(x) = \gamma\sqrt{1+x^2}$, with $\theta = 2$ and $\gamma = \sqrt{1/2}$, $I_1 = [-0.9, 0.8]$.
- (2) Hyperbolic tangent of an Ornstein-Uhlenbeck,

$$b_2(x) = (1-x^2) \left(-\frac{r}{2} \operatorname{atanh}(x) - \frac{\gamma^2}{4} x \right), \quad \sigma_2(x) = \frac{\gamma}{2} (1-x^2), \quad r = 2, \quad \gamma = 2, \quad I_2 = [-0.9, 0.9].$$

- (3) Exponential of an Ornstein-Uhlenbeck,

$$b_3(x) = x \left(-\frac{r}{2} \log(x^+) + \frac{\gamma^2}{8} \right), \quad \sigma_3(x) = \frac{\gamma}{2} x^+, \quad r = 1, \quad \gamma = 2, \quad I_3 = [0.44, 2].$$

- (4) $X_t = G_1(\xi_t)$ with $d\xi_t = \alpha(\xi_t)dt + dW_t$, $\alpha(x) = -\theta x/\sqrt{1+c^2x^2}$, $G_1(x) = \operatorname{asinh}(c\xi_t)$, $\theta = 3$, $c = 2$, $I_4 = [-1.15, 1.15]$, leading to

$$b_4(x) = - \left(\theta + \frac{c^2}{2 \cosh(x)} \right) \frac{\sinh(x)}{\cosh^2(x)}, \quad \sigma_4(x) = \frac{c}{\cosh(x)}.$$

- (5) $X_t = G_2(\xi_t)$ with ξ as previously and $G_2(x) = \operatorname{asinh}(x-5) + \operatorname{asinh}(x+5)$, leading to

$$b_5(x) = G_2'(H(x))\alpha(H(x)) + \frac{1}{2}G_2''(H(x)), \quad \sigma_5(x) = \frac{1}{\sqrt{1+(H(x)-5)^2}} + \frac{1}{\sqrt{1+(H(x)+5)^2}},$$

where $\theta = 1$, $c = 10$ (in the definition of α), $I_5 = [-4, 4]$ and

$$H(x) = G_2^{-1}(x) = \frac{1}{\sqrt{2} \sinh(x)} [(49 + \cosh(x)) \sinh(x)^2 + 100(1 - \cosh(x))]^{1/2}.$$

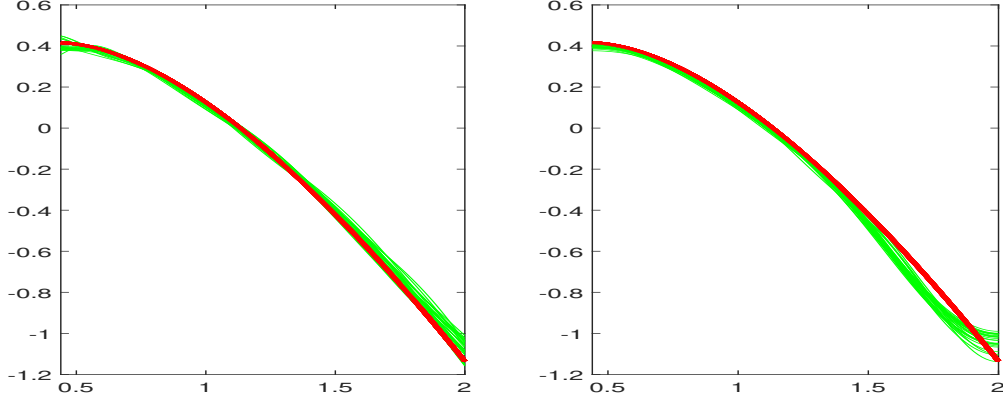


FIGURE 1. Example 3. True functions in bold red and beam of 25 estimated drift b_3 with Hermite (left) and cosine (right) bases, $\rho = 0.5$. The $\text{MISE} \times 100$ are 0.12, 0.33 and the mean of selected dimensions are 8.4, 4.3.

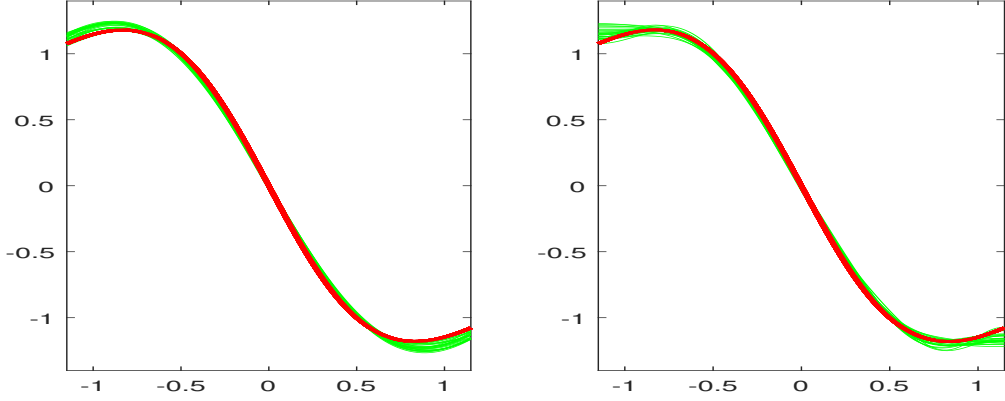


FIGURE 2. Example 4. True functions in bold red and beam of 25 estimated drift b_4 with Hermite (left) and cosine (right) bases, $\rho = 0.5$. The $\text{MISE} \times 100$ are 0.36, 0.26 and the mean of selected dimensions are 4.4, 6.0.

Models 1, 4, 5 are simulated by Euler scheme with step Δ , directly for X in example 1 or for ξ in examples 4 and 5, with transformations G_1 and G_2 in a second stage. The underlying Ornstein-Uhlenbeck processes in models 2 and 3 are generated by exact autoregressive scheme with step Δ . Details can be found in Comte and Genon-Catalot [8] for examples 1, 2, 3 and in Comte et al. [10] for examples 4 and 5. The dependency is contained in the Toeplitz variance matrix¹ $R(\rho) := (\rho^{|i-j|})_{1 \leq i, j \leq N}$ for different values of ρ . The choice $\rho = 0$ corresponds to the independent case, and we also experiment $\rho = 0.5$ (mild dependency) and $\rho = 0.9$ (strong correlations). Assumption 3.4 is not fulfilled but we can consider that the coefficients are in fact null when $|i - j|$ is large enough. The orders of some quantities related to $R(\rho)$ are given in Table 1, and clearly, $\sum_{i,k} R(\rho)_{i,k}/N$ and $\|R(\rho)\|_{\text{op}}$ are very close. The penalty term is computed as in Comte and Genon-Catalot [7], by an empirical version which directly takes dependence into account

¹This matrix is indeed a correlation matrix as it is the variance of a stationary AR(1) process, $X_t = \rho X_{t-1} + \varepsilon_t$, for i.i.d. centered ε_t 's.

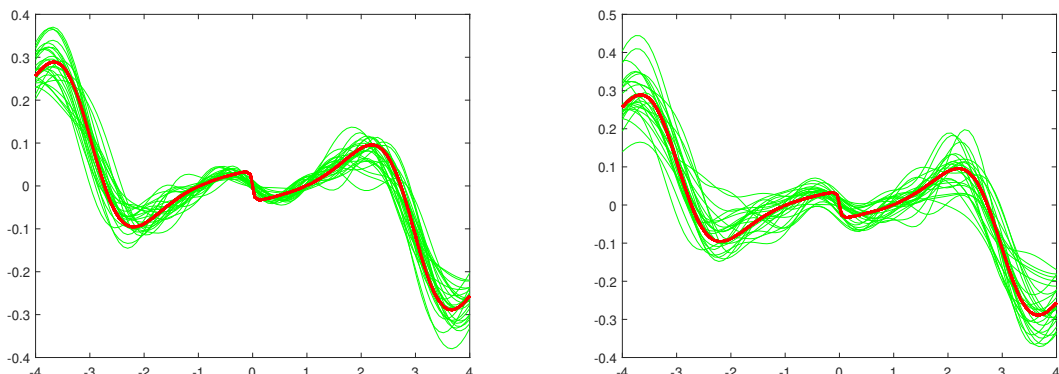


FIGURE 3. Example 5. True functions in bold red and beam of 25 estimated drift b_5 with Hermite basis for $\rho = 0$ (left) and $\rho = 0.9$ (right). The $\text{MISE} \times 100$ are 0.87, 1.67 and the mean of selected dimensions are 12.5 in both cases.

	$\rho = 0$	$\rho = 0.5$	$\rho = 0.9$
$N^{-1} \sum_{i,j} R_{i,j}(\rho)$	1	2.96	17.2
$\ R(\rho)\ _{\text{op}}$	1	2.99	17.9

TABLE 1. Order of different quantities associated to the matrix $R(\rho)$.

without requiring any information on R :

$$\widehat{\text{pen}}(m) := \kappa \frac{m}{NT} \|\widehat{\Psi}_m^{-1} \widehat{\Psi}_{m,\sigma}\|_{\text{op}} \quad \text{with} \quad \widehat{\Psi}_{m,\sigma} = ((\sigma\varphi_j, \sigma\varphi_\ell)_N)_{j,\ell}$$

and $\kappa = 2$ for both bases. Then, the model \widehat{m} is chosen as the minimizer of $-\|\widehat{b}_m\|_N^2 + \widehat{\text{pen}}(m)$ for $m \leq 10$ (resp. $m \leq 20$) for the Hermite basis, except in example 5, where we set $m \leq 15$ because otherwise the selected dimension was systematically the maximal one (resp. for the cosine basis), such that

$$m \|\widehat{\Psi}_m^{-1}\|_{\text{op}}^{1/4} \leq NT \quad (\text{empirical collection of models}).$$

We present illustrations of the estimation procedure obtained from simulated paths in Figures 1, 2, 3 for examples 3, 4 and 5. Figures 1 and 2 allow the comparison of the results obtained for Hermite (left pictures) and cosine (right pictures) bases, for $\rho = 0.5$. Figure 3 shows the difference of estimation in the Hermite basis when $\rho = 0$ (left picture) and when $\rho = 0.9$ (right picture). The scenario is the same in the three figures: $N = 200$ and $T = 100$ (with 1000 observations with step $\Delta = 0.1$ for each path). The MISE over the 25 repetitions are given, together with the mean of the selected dimensions. We can see that the examples are quite different, and that the estimation method works in a convincing way, even for strong dependency ($\rho = 0.9$).

We also illustrate on the scenario $N = 100$ and $T = 100$ (with 1000 observations with step $\Delta = 0.1$) for each path, which was a middle scenario in Comte and Genon-Catalot [7], the influence of the value of ρ on the MISE computed over 200 repetitions: the results are given in Table 2. We see that the MISE increases when ρ increases, slightly from $\rho = 0$ to $\rho = 0.5$ and much more importantly from $\rho = 0.5$ to $\rho = 0.9$. On the contrary, the selected dimensions for each basis are rather unchanged in these different cases. This suggests that bias and variance increase simultaneously and proportionally. The Hermite basis gives lower MISEs for examples 1 to 3, and the cosine basis wins for examples 4 and 5.

Ex.		$\rho = 0$		$\rho = 0.5$		$\rho = 0.9$	
		Hermite	Cosine	Herm	Cos	Herm	Cos
Ex. 1	MISE	0.11 _(0.08)	0.80 _(0.24)	0.13 _(0.10)	0.83 _(0.29)	0.63 _(0.54)	1.38 _(6.29)
	Dim	6.2 _(0.8)	6.2 _(1.5)	6.1 _(0.5)	6.2 _(1.5)	6.3 _(1.1)	6.3 _(1.2)
Ex. 2	MISE	0.78 _(0.18)	0.95 _(0.19)	0.78 _(0.18)	0.94 _(0.18)	1.02 _(0.47)	1.18 _(0.49)
	Dim	6.1 _(0.5)	10.3 _(1.9)	6.1 _(0.5)	10.4 _(2.0)	6.1 _(0.5)	10.3 _(2.0)
Ex. 3	MISE	0.22 _(0.16)	0.34 _(0.11)	0.21 _(0.16)	0.37 _(0.14)	0.44 _(0.41)	0.55 _(0.43)
	Dim	7.8 _(0.7)	4.1 _(0.4)	7.7 _(0.7)	4.1 _(0.4)	7.8 _(0.8)	4.2 _(0.6)
Ex. 4	MISE	0.41 _(0.18)	0.35 _(0.15)	0.46 _(0.21)	0.39 _(0.18)	1.06 _(0.66)	1.02 _(0.60)
	Dim	4.4 _(0.8)	5.1 _(1.4)	4.4 _(0.8)	5.2 _(1.3)	4.9 _(1.3)	5.6 _(1.5)
Ex. 5	MISE	1.55 _(0.85)	1.49 _(0.61)	1.81 _(0.97)	1.58 _(0.62)	4.14 _(4.22)	3.15 _(3.55)
	Dim	11.2 _(1.3)	6.3 _(0.8)	11.4 _(1.4)	6.2 _(0.6)	11.5 _(1.5)	6.3 _(0.8)

TABLE 2. 100 MISE (with 100 StD in parenthesis) and mean selected dimensions (with StD in parenthesis) for the examples 1 to 5, $N = 100$ and $T = 100$, for Hermite and cosine bases and 3 values of ρ (0 for independence, $\rho = 0.9$ for strong dependency).

6. PROOFS

6.1. Proof of Lemma 3.2. First of all, let us show that the symmetric matrix $\Psi_{m,\sigma}$ is nonnegative. Indeed, for every $y \in \mathbb{R}^m$,

$$\begin{aligned} y^* \Psi_{m,\sigma} y &= \frac{1}{NT} \sum_{j,\ell=1}^m y_j y_\ell \sum_{i,k=1}^N \mathbb{E} \left(\left(\int_0^T \sigma(X_s^i) \varphi_j(X_s^i) dB_s^i \right) \left(\int_0^T \sigma(X_s^k) \varphi_\ell(X_s^k) dB_s^k \right) \right) \\ &= \frac{1}{NT} \mathbb{E} \left[\left(\sum_{i=1}^N \int_0^T \sigma(X_s^i) \tau_y(X_s^i) dB_s^i \right)^2 \right] \geq 0 \quad \text{with} \quad \tau_y(\cdot) := \sum_{j=1}^m y_j \varphi_j(\cdot). \end{aligned}$$

On the one hand, since $\Psi_{m,\sigma}$ is a nonnegative matrix, since $d\langle B^i, B^k \rangle_t = R_{i,k} dt$ for every $i, k \in \{1, \dots, N\}$, and by the stochastic integration by parts formula,

$$\begin{aligned} \text{trace}(\Psi_m^{-1} \Psi_{m,\sigma}) &\leq \|\Psi_m^{-1}\|_{\text{op}} \text{trace}(\Psi_{m,\sigma}) = \frac{1}{NT} \|\Psi_m^{-1}\|_{\text{op}} \sum_{j=1}^m \mathbb{E} \left[\left(\sum_{i=1}^N \int_0^T \sigma(X_s^i) \varphi_j(X_s^i) dB_s^i \right)^2 \right] \\ &= \frac{1}{NT} \|\Psi_m^{-1}\|_{\text{op}} \sum_{j=1}^m \sum_{i,k=1}^N \int_0^T R_{i,k} \mathbb{E}(\sigma(X_s^i) \varphi_j(X_s^i) \sigma(X_s^k) \varphi_j(X_s^k)) ds \\ &\leq \frac{1}{N} \|\Psi_m^{-1}\|_{\text{op}} \left(N + \sum_{i \neq k} |R_{i,k}| \right) \sum_{j=1}^m \int_{-\infty}^{\infty} \sigma(x)^2 \varphi_j(x)^2 f_T(x) dx \\ &\leq \mathbf{c}_1 \|\Psi_m^{-1}\|_{\text{op}} L(m) \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right) \quad \text{with} \quad \mathbf{c}_1 = \int_{-\infty}^{\infty} \sigma(x)^2 f_T(x) dx. \end{aligned}$$

On the other hand, assume now that σ is bounded. Again, since $d\langle B^i, B^k \rangle_t = R_{i,k} dt$ for every $i, k \in \{1, \dots, N\}$, and by the stochastic integration by parts formula, for every $y \in \mathbb{R}^m$,

$$\begin{aligned}
y^* \Psi_{m,\sigma} y &= \frac{1}{NT} \mathbb{E} \left[\left(\sum_{i=1}^N \int_0^T \sigma(X_s^i) \tau_y(X_s^i) dB_s^i \right)^2 \right] \\
&\leq \frac{1}{T} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right) \int_0^T \mathbb{E}(\sigma(X_s)^2 \tau_y(X_s)^2) ds \\
&\leq \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right) \int_{-\infty}^{\infty} \sigma(x)^2 \left(\sum_{j=1}^m y_j \varphi_j(x) \right)^2 f_T(x) dx \\
(8) \quad &\leq \|\sigma\|_{\infty}^2 \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right) \|\Psi_m^{1/2} y\|_{2,m}^2.
\end{aligned}$$

Thus, since $\Psi_{m,\sigma}$ is nonnegative, and by Inequality (8),

$$\begin{aligned}
\text{trace}(\Psi_m^{-1/2} \Psi_{m,\sigma} \Psi_m^{-1/2}) &\leq m \|\Psi_m^{-1/2} \Psi_{m,\sigma} \Psi_m^{-1/2}\|_{\text{op}} \\
&= m \cdot \sup\{y^* \Psi_{m,\sigma} y ; y \in \mathbb{R}^m \text{ and } \|\Psi_m^{1/2} y\|_{2,m} = 1\} \\
&\leq m \|\sigma\|_{\infty}^2 \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right).
\end{aligned}$$

6.2. Proof of Theorem 3.5. The proof of Theorem 3.5 relies on the two following lemmas.

Lemma 6.1. *There exists a deterministic constant $\mathfrak{c}_{6.1} > 0$, not depending on m and N , such that*

$$\mathbb{E}(|\widehat{\mathbf{E}}_m^* \widehat{\mathbf{E}}_m|^2) \leq \mathfrak{c}_{6.1} \frac{mL(m)^2}{N^2} \left[1 + \left(\frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right)^2 \right].$$

Lemma 6.2. *Consider the event*

$$\Omega_m := \left\{ \sup_{\tau \in \mathcal{S}_m} \left| \frac{\|\tau\|_N^2}{\|\tau\|_{f_T}^2} - 1 \right| \leq \frac{1}{2} \right\}.$$

Under Assumptions 3.1, 3.3 and 3.4, there exists a deterministic constant $\mathfrak{c}_{6.2} > 0$, not depending on m and N , such that

$$\mathbb{P}(\Omega_m^c) \leq \frac{\mathfrak{c}_{6.2}}{N^6} \quad \text{and} \quad \mathbb{P}(\Lambda_m^c) \leq \frac{\mathfrak{c}_{6.2}}{N^6}.$$

6.2.1. Steps of the proof. First of all,

$$\begin{aligned}
\|\widehat{b}_m - b_I\|_N^2 &= \|b_I\|_N^2 \mathbf{1}_{\Lambda_m^c} + \|\widehat{b}_m - b_I\|_N^2 \mathbf{1}_{\Lambda_m} \\
(9) \quad &= U_1 + U_2 + U_3
\end{aligned}$$

where $U_1 := \|b_I\|_N^2 \mathbf{1}_{\Lambda_m^c}$,

$$U_2 := \|\widehat{b}_m - b_I\|_N^2 \mathbf{1}_{\Lambda_m \cap \Omega_m} \quad \text{and} \quad U_3 := \|\widehat{b}_m - b_I\|_N^2 \mathbf{1}_{\Lambda_m \cap \Omega_m^c}.$$

Let us find suitable bounds on $\mathbb{E}(U_1)$, $\mathbb{E}(U_2)$ and $\mathbb{E}(U_3)$.

- **Bound on $\mathbb{E}(U_1)$.** By Cauchy-Schwarz's inequality,

$$\begin{aligned}
\mathbb{E}(U_1) &\leq \mathbb{E}(\|b_I\|_N^4)^{1/2} \mathbb{P}(\Lambda_m^c)^{1/2} \leq \mathbb{E} \left(\frac{1}{T} \int_0^T b_I(X_t)^4 dt \right)^{1/2} \mathbb{P}(\Lambda_m^c)^{1/2} \\
&\leq \mathfrak{c}_1 \mathbb{P}(\Lambda_m^c)^{1/2} < \infty \quad \text{with} \quad \mathfrak{c}_1 = \left(\int_{-\infty}^{\infty} b_I(x)^4 f_T(x) dx \right)^{1/2} < \infty.
\end{aligned}$$

- **Bound on $\mathbb{E}(U_2)$.** Let $\Pi_{N,m}(\cdot)$ be the orthogonal projection from $\mathbb{L}^2(I, f_T(x)dx)$ onto \mathcal{S}_m with respect to the empirical scalar product $\langle \cdot, \cdot \rangle_N$. Then,

$$(10) \quad \|\widehat{b}_m - b_I\|_N^2 = \|\widehat{b}_m - \Pi_{N,m}(b_I)\|_N^2 + \min_{\tau \in \mathcal{S}_m} \|b_I - \tau\|_N^2.$$

As in the proof of Comte and Genon-Catalot [7], Proposition 2.1, on Ω_m ,

$$\|\widehat{b}_m - \Pi_{N,m}(b_I)\|_N^2 = \widehat{\mathbf{E}}_m^* \widehat{\Psi}_m^{-1} \widehat{\mathbf{E}}_m \leq 2\widehat{\mathbf{E}}_m^* \Psi_m^{-1} \widehat{\mathbf{E}}_m.$$

So,

$$\begin{aligned} \mathbb{E}(\|\widehat{b}_m - \Pi_{N,m}(b_I)\|_N^2 \mathbf{1}_{\Lambda_m \cap \Omega_m}) &\leq 2\mathbb{E} \left(\sum_{j,\ell=1}^m [\widehat{\mathbf{E}}_m]_j [\widehat{\mathbf{E}}_m]_\ell \Psi_m^{-1}(j, \ell) \right) \\ &= \frac{2}{NT} \sum_{j,\ell=1}^m \Psi_{m,\sigma}(j, \ell) \Psi_m^{-1}(j, \ell) = \frac{2}{NT} \text{trace}(\Psi_m^{-1} \Psi_{m,\sigma}) \\ &= \frac{2}{NT} \text{trace}(\Psi_m^{-1/2} \Psi_{m,\sigma} \Psi_m^{-1/2}). \end{aligned}$$

Then, by Equality (10) and Lemma 3.2,

$$\begin{aligned} \mathbb{E}(U_2) &\leq \mathbb{E} \left(\min_{\tau \in \mathcal{S}_m} \|b_I - \tau\|_N^2 \right) + \frac{2}{NT} \text{trace}(\Psi_m^{-1/2} \Psi_{m,\sigma} \Psi_m^{-1/2}) \\ &\leq \min_{\tau \in \mathcal{S}_m} \|b_I - \tau\|_{f_T}^2 + \frac{2m}{NT} \|\sigma\|_\infty^2 \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right). \end{aligned}$$

- **Bound on $\mathbb{E}(U_3)$.** By the definition of the event Λ_m and by Lemma 6.1,

$$\begin{aligned} \mathbb{E}(\|\widehat{b}_m - \Pi_{N,m}(b_I)\|_N^2 \mathbf{1}_{\Lambda_m \cap \Omega_m^c}) &= \mathbb{E}[(\widehat{\mathbf{E}}_m^* \widehat{\Psi}_m^{-1} \widehat{\mathbf{E}}_m) \mathbf{1}_{\Lambda_m \cap \Omega_m^c}] \\ &\leq \mathbb{E}(\|\widehat{\Psi}_m^{-1}\|_{\text{op}} |\widehat{\mathbf{E}}_m^* \widehat{\mathbf{E}}_m| \mathbf{1}_{\Lambda_m \cap \Omega_m^c}) \\ &\leq \frac{\mathbf{c}_T(p)}{L(m)} \cdot \frac{NT}{\log(NT)} \mathbb{E}(|\widehat{\mathbf{E}}_m^* \widehat{\mathbf{E}}_m|^2)^{1/2} \mathbb{P}(\Omega_m^c)^{1/2} \\ &\leq \frac{\mathbf{c}_2 m^{1/2}}{\log(NT)} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right) \mathbb{P}(\Omega_m^c)^{1/2} \end{aligned}$$

where $\mathbf{c}_2 > 0$ is a deterministic constant not depending on m and N . Then,

$$\begin{aligned} \mathbb{E}(U_3) &\leq \mathbb{E}(\|\widehat{b}_m - \Pi_{N,m}(b_I)\|_N^2 \mathbf{1}_{\Lambda_m \cap \Omega_m^c}) + \mathbb{E}(\|b_I\|_N^2 \mathbf{1}_{\Lambda_m \cap \Omega_m^c}) \\ &\leq \frac{\mathbf{c}_2 m^{1/2}}{\log(NT)} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right) \mathbb{P}(\Omega_m^c)^{1/2} + \mathbf{c}_1 \mathbb{P}(\Omega_m^c)^{1/2}. \end{aligned}$$

So,

$$\begin{aligned} \mathbb{E}(\|\widetilde{b}_m - b_I\|_N^2) &\leq \min_{\tau \in \mathcal{S}_m} \|b_I - \tau\|_{f_T}^2 \\ &\quad + \left[\frac{2m}{NT} \|\sigma\|_\infty^2 + \mathbf{c}_2 \frac{\sqrt{m \mathbb{P}(\Omega_m^c)}}{\log(NT)} \right] \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right) + \mathbf{c}_1 (\mathbb{P}(\Lambda_m^c)^{1/2} + \mathbb{P}(\Omega_m^c)^{1/2}). \end{aligned}$$

Therefore, by Lemma 6.2, there exists a deterministic constant $\mathbf{c}_3 > 0$, not depending on m and N , such that

$$\mathbb{E}(\|\widetilde{b}_m - b_I\|_N^2) \leq \min_{\tau \in \mathcal{S}_m} \|b_I - \tau\|_{f_T}^2 + \frac{\mathbf{c}_3 m}{NT} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right) + \frac{\mathbf{c}_3}{N}.$$

6.2.2. *Proof of Lemma 6.1.* By Jensen's inequality, by Burkholder-Davis-Gundy's inequality, and since $d\langle B^i, B^k \rangle_t = R_{i,k} dt$ for every $i, k \in \{1, \dots, N\}$, there exists a deterministic constant $\mathbf{c}_1 > 0$, not depending on m and N , such that

$$\begin{aligned} \mathbb{E}(|\widehat{\mathbf{E}}_m^* \widehat{\mathbf{E}}_m|^2) &\leq m \sum_{j=1}^m \mathbb{E}(\widehat{\mathbf{E}}_m(j)^4) \leq \frac{\mathbf{c}_1 m}{N^4 T^4} \sum_{j=1}^m \mathbb{E} \left(\left\langle \sum_{i=1}^N \int_0^T \sigma(X_s^i) \varphi_j(X_s^i) dB_s^i \right\rangle_T^2 \right) \\ &\leq \frac{2\mathbf{c}_1 m}{N^4 T^4} \sum_{j=1}^m (\mathbb{E}(D_j^2) + \mathbb{E}(A_j^2)), \end{aligned}$$

where

$$D_j := \sum_{i=1}^N \int_0^T \sigma(X_s^i)^2 \varphi_j(X_s^i)^2 ds \quad \text{and} \quad A_j := \sum_{i \neq k} R_{i,k} \int_0^T \sigma(X_s^i) \varphi_j(X_s^i) \sigma(X_s^k) \varphi_j(X_s^k) ds$$

for every $j \in \{1, \dots, m\}$. On the one hand, by Jensen's inequality,

$$\begin{aligned} \sum_{j=1}^m \mathbb{E}(D_j^2) &\leq NT \sum_{j=1}^m \sum_{i=1}^N \int_0^T \mathbb{E}(\sigma(X_s^i)^4 \varphi_j(X_s^i)^4) ds \\ &\leq N^2 T L(m)^2 \int_0^T \mathbb{E}(\sigma(X_s)^4) ds = N^2 T^2 L(m)^2 \int_{-\infty}^{\infty} \sigma(x)^4 f_T(x) dx. \end{aligned}$$

On the other hand, by Jensen's inequality and Cauchy-Schwarz's inequality,

$$\begin{aligned} \sum_{j=1}^m \mathbb{E}(A_j^2) &\leq T \left(\sum_{i \neq k} |R_{i,k}| \right) \sum_{j=1}^m \sum_{i \neq k} |R_{i,k}| \int_0^T \mathbb{E}(\sigma(X_s^i)^2 \varphi_j(X_s^i)^2 \sigma(X_s^k)^2 \varphi_j(X_s^k)^2) ds \\ &\leq T \left(\sum_{i \neq k} |R_{i,k}| \right)^2 \sum_{j=1}^m \int_0^T \mathbb{E}(\sigma(X_s)^4 \varphi_j(X_s)^4) ds \\ &\leq T^2 \left(\sum_{i \neq k} |R_{i,k}| \right)^2 L(m)^2 \int_{-\infty}^{\infty} \sigma(x)^4 f_T(x) dx. \end{aligned}$$

Therefore,

$$\mathbb{E}(|\widehat{\mathbf{E}}_m^* \widehat{\mathbf{E}}_m|^2) \leq \frac{\mathbf{c}_2}{N^2 T^2} mL(m)^2 \left[1 + \left(\frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right)^2 \right]$$

with

$$\mathbf{c}_2 = 2\mathbf{c}_1 \int_{-\infty}^{\infty} \sigma(x)^4 f_T(x) dx.$$

6.2.3. *Proof of Lemma 6.2.* Let $(\overline{\varphi}_1, \dots, \overline{\varphi}_{N_T})$ be the orthonormal family of $L^2(I, f_T(x) dx)$ derived from $(\varphi_1, \dots, \varphi_{N_T})$ via Gram-Schmidt's method. Consider also the matrix

$$\widehat{\mathbf{G}}_m := \sum_{i=1}^N \widehat{\mathbf{G}}_m(X^i),$$

where

$$\widehat{\mathbf{G}}_m(\psi) := \frac{1}{NT} \left(\int_0^T \overline{\varphi}_j(\psi(t)) \overline{\varphi}_\ell(\psi(t)) dt \right)_{j, \ell \in \{1, \dots, m\}}; \quad \forall \psi \in \Omega.$$

The random matrix $\widehat{\mathbf{G}}_m(X^i)$ has the same eigenvalues as $N^{-1}\Psi_m^{-1/2}\widehat{\Psi}_m(X^i)\Psi_m^{-1/2}$, where

$$\widehat{\Psi}_m(\psi) := \left(\frac{1}{T} \int_0^T \varphi_j(\psi(t))\varphi_\ell(\psi(t))dt \right)_{j,\ell \in \{1,\dots,m\}} ; \forall \psi \in \Omega.$$

Moreover, for every $\psi \in \Omega$, by Jensen's and Cauchy-Schwarz's inequalities,

$$\begin{aligned} \|\widehat{\Psi}_m(\psi)\|_{\text{op}}^2 &= \sup_{\|x\|_{2,m}=1} \sum_{\ell=1}^m \left[\frac{1}{T} \int_0^T \left(\sum_{j=1}^m \varphi_j(\psi(t))\varphi_\ell(\psi(t))x_j \right) dt \right]^2 \\ &\leq \frac{1}{T} \sup_{\|x\|_{2,m}=1} \sum_{\ell=1}^m \int_0^T \varphi_\ell(\psi(t))^2 \left(\sum_{j=1}^m \varphi_j(\psi(t))x_j \right)^2 dt \\ (11) \quad &\leq \frac{L(m)}{T} \sup_{\|x\|_{2,m}=1} \int_0^T \left(\sum_{j=1}^m \varphi_j(\psi(t))x_j \right)^2 dt \leq L(m)^2. \end{aligned}$$

Notations:

- The semidefinite order on symmetric matrices is denoted by \preceq .
- $\mathbb{E}_0(\cdot) := \mathbb{E}(\cdot)$ and $\mathbb{E}_i(\cdot) := \mathbb{E}(\cdot | \mathcal{F}_i)$ for every $i \in \{1, \dots, N\}$.

First of all, note that

$$\|\widehat{\mathbf{G}}_m - \mathbf{I}\|_{\text{op}} \leq M_N + R_N,$$

where

$$M_N := \left\| \sum_{i=1}^N (\widehat{\mathbf{G}}_m(X^i) - \mathbb{E}_{i-1}(\widehat{\mathbf{G}}_m(X^i))) \right\|_{\text{op}} \quad \text{and} \quad R_N := \left\| \sum_{i=1}^N (\mathbb{E}_{i-1}(\widehat{\mathbf{G}}_m(X^i)) - N^{-1}\mathbf{I}) \right\|_{\text{op}}.$$

The proof of Lemma 6.2 is dissected in three steps. Step 1 deals with a suitable bound on $\mathbb{P}(M_N > \delta/2)$, $\delta > 0$, step 2 with a suitable bound on $\mathbb{P}(R_N > \delta/2)$, and the conclusion comes in step 3.

Step 1. For any $\delta > 0$, let us establish a suitable bound on $\mathbb{P}(M_N > \delta)$. For every $i \in \{1, \dots, N\}$, since

$$\widetilde{\mathbf{G}}_m(X^i) := \widehat{\mathbf{G}}_m(X^i) - \mathbb{E}_{i-1}(\widehat{\mathbf{G}}_m(X^i))$$

is a symmetric matrix, by Jensen's inequality and by Inequality (11),

$$\begin{aligned} \widetilde{\mathbf{G}}_m(X^i)^2 &\preceq \lambda_{\max}[\widetilde{\mathbf{G}}_m(X^i)^2]\mathbf{I} \\ &= \|\widehat{\mathbf{G}}_m(X^i) - \mathbb{E}_{i-1}(\widehat{\mathbf{G}}_m(X^i))\|_{\text{op}}^2 \mathbf{I} \\ &\preceq \frac{2}{N^2} (\|\Psi_m^{-1/2}\widehat{\Psi}_m(X^i)\Psi_m^{-1/2}\|_{\text{op}}^2 + \mathbb{E}_{i-1}(\|\Psi_m^{-1/2}\widehat{\Psi}_m(X^i)\Psi_m^{-1/2}\|_{\text{op}}^2)) \mathbf{I} \\ &\preceq \frac{2}{N^2} (\|\widehat{\Psi}_m(X^i)\|_{\text{op}}^2 + \mathbb{E}_{i-1}(\|\widehat{\Psi}_m(X^i)\|_{\text{op}}^2)) \|\Psi_m^{-1}\|_{\text{op}}^2 \mathbf{I} \preceq \mathbf{A}_i^2 \end{aligned}$$

with

$$\mathbf{A}_i^2 := \frac{4}{N^2} [L(m)(\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^2 \mathbf{I}.$$

So, by Azuma's inequality for matrix martingales (see Tropp [22], Theorem 7.1),

$$\mathbb{P} \left(\left\| \sum_{i=1}^N \widetilde{\mathbf{G}}_m(X^i) \right\|_{\text{op}} > \delta \right) \leq m \exp \left(-\frac{\delta^2}{8\sigma^2} \right)$$

with

$$\sigma^2 = \left\| \sum_{i=1}^N \mathbf{A}_i^2 \right\|_{\text{op}} = \frac{4}{N} [L(m)(\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^2.$$

This leads to

$$\mathbb{P}(M_N > \delta) \leq m \exp \left[-\frac{\delta^2 N}{32[L(m)(\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^2} \right].$$

Step 2. For any $\delta > 0$, let us establish a suitable bound on $\mathbb{P}(R_N > \delta)$. First of all, let us recall that under Assumption 3.4,

$$\mathcal{I}_N = \{i \in \{1, \dots, N\} : B^i \text{ is not independent of } \mathcal{F}_{i-1}\}.$$

For every $i \in \{1, \dots, N\} \setminus \mathcal{I}_N$, since $(\bar{\varphi}_1, \dots, \bar{\varphi}_{N_T})$ is an orthonormal family of $\mathbb{L}^2(I, f_T(x)dx)$,

$$\begin{aligned} \mathbb{E}_{i-1}(\widehat{\mathbf{G}}_m(X^i)) &= \mathbb{E}(\widehat{\mathbf{G}}_m(X)) = \left(\frac{1}{NT} \int_0^T \mathbb{E}(\bar{\varphi}_j(X_t)\bar{\varphi}_\ell(X_t)) dt \right)_{j,\ell} \\ &= \frac{1}{N} (\langle \bar{\varphi}_j, \bar{\varphi}_\ell \rangle_{f_T})_{j,\ell} = \frac{1}{N} \mathbf{I}. \end{aligned}$$

Then,

$$R_N = \left\| \sum_{i \in \mathcal{I}_N} (\mathbb{E}_{i-1}(\widehat{\mathbf{G}}_m(X^i)) - N^{-1} \mathbf{I}) \right\|_{\text{op}}.$$

By Markov's inequality and Jensen's inequality (usual and conditional),

$$\begin{aligned} \mathbb{P}(R_N > \delta) &\leq \frac{\mathbb{E}(R_N^p)}{\delta^p} \leq \frac{|\mathcal{I}_N|^{p-1}}{\delta^p} \sum_{i \in \mathcal{I}_N} \mathbb{E}(\|\mathbb{E}_{i-1}(\widehat{\mathbf{G}}_m(X^i)) - N^{-1} \mathbf{I}\|_{\text{op}}^p) \\ &\leq \frac{|\mathcal{I}_N|^p}{\delta^p} \mathbb{E}(\|\widehat{\mathbf{G}}_m(X) - N^{-1} \mathbf{I}\|_{\text{op}}^p) \\ &= \frac{|\mathcal{I}_N|^p}{\delta^p N^p} \mathbb{E}(\|\Psi_m^{-1/2} \widehat{\Psi}_m(X) \Psi_m^{-1/2} - \mathbf{I}\|_{\text{op}}^p) \\ &\leq \frac{2^{p-1} |\mathcal{I}_N|^p}{\delta^p N^p} [\mathbb{E}(\|\widehat{\Psi}_m(X)\|_{\text{op}}^p) \|\Psi_m^{-1}\|_{\text{op}}^p + 1] \leq \frac{2^p |\mathcal{I}_N|^p}{\delta^p N^p} [L(m)(\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^p. \end{aligned}$$

Step 3 (conclusion). For any $\delta > 0$, the two previous steps leads to

$$\begin{aligned} \mathbb{P}(\|\widehat{\mathbf{G}}_m - \mathbf{I}\|_{\text{op}} > \delta) &\leq \mathbb{P} \left(\left\{ M_N > \frac{\delta}{2} \right\} \cup \left\{ R_N > \frac{\delta}{2} \right\} \right) \\ &\leq m \exp \left[-\frac{\delta^2 N}{128[L(m)(\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^2} \right] \\ &\quad + \frac{2^{2p} |\mathcal{I}_N|^p}{\delta^p N^p} [L(m)(\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^p. \end{aligned} \tag{12}$$

As established in the beginning of the proof of Comte and Genon-Catalot [7], Proposition 2.1,

$$\Omega_m = \left\{ \sup_{\tau \in \mathcal{S}_m} \left| \frac{\|\tau\|_N^2}{\|\tau\|_{f_T}^2} - 1 \right| \leq \frac{1}{2} \right\} = \left\{ \|\widehat{\mathbf{G}}_m - \mathbf{I}\|_{\text{op}} \leq \frac{1}{2} \right\}.$$

Then, by Inequality (12), by Assumptions 3.3 and 3.4 (leading to (7)), and since $p \geq 12$,

$$\begin{aligned} \mathbb{P}(\Omega_m^c) &\leq m \exp \left(-\frac{\log(NT)}{512 \mathbf{c}_T(p) T} \right) + \frac{2^{3p} (\mathbf{c}_T(p) T)^{p/2}}{2^{p/2}} \cdot \frac{|\mathcal{I}_N|^p}{N^{p/2} \log(NT)^{p/2}} \\ &\leq \mathbf{c}_1 \left(\frac{m}{N^{1+p/2}} + \frac{|\mathcal{I}_N|^p}{N^{p/2} \log(NT)^{p/2}} \right) \leq \frac{\mathbf{c}_1 (1 + \mathbf{c}_{3.4}(p))}{N^6} \end{aligned}$$

where $\mathbf{c}_1 > 0$ is a deterministic constant not depending on m and N . Moreover, on Ω_m^c and by Assumption 3.3,

$$[L(m)(\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^2 \leq \frac{\mathbf{c}_T(p)}{2} \cdot \frac{NT}{\log(NT)} \quad \text{and} \quad L(m)(\|\widehat{\Psi}_m^{-1}\|_{\text{op}} \vee 1) > \mathbf{c}_T(p) \frac{NT}{\log(NT)}.$$

The first inequality implies that

$$L(m)\|\Psi_m^{-1}\|_{\text{op}} \leq \frac{\mathfrak{c}_T(p)}{2} \cdot \frac{NT}{\log(NT)} \quad \text{and} \quad L(m) \leq \frac{\mathfrak{c}_T(p)}{2} \cdot \frac{NT}{\log(NT)},$$

and then the second one leads to

$$\begin{aligned} \mathfrak{c}_T(p) \frac{NT}{\log(NT)} &< L(m)\|\widehat{\Psi}_m^{-1}\|_{\text{op}} \leq L(m)(\|\widehat{\Psi}_m^{-1} - \Psi_m^{-1}\|_{\text{op}} + \|\Psi_m^{-1}\|_{\text{op}}) \\ &\leq L(m)\|\widehat{\Psi}_m^{-1} - \Psi_m^{-1}\|_{\text{op}} + \frac{\mathfrak{c}_T(p)}{2} \cdot \frac{NT}{\log(NT)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}(\Lambda_m^c) &\leq \mathbb{P}\left(\frac{\mathfrak{c}_T(p)}{2} \cdot \frac{NT}{\log(NT)} \leq L(m)\|\widehat{\Psi}_m^{-1} - \Psi_m^{-1}\|_{\text{op}}\right) \\ &\leq \mathbb{P}(\|\Psi_m^{-1}\|_{\text{op}} < \|\widehat{\Psi}_m^{-1} - \Psi_m^{-1}\|_{\text{op}}) \leq \mathbb{P}(\Omega_m^c) \leq \frac{\mathfrak{c}_1(1 + \mathfrak{c}_{3.4}(p))}{N^6}. \end{aligned}$$

6.3. Proof of Theorem 4.3. Let us consider the events

$$\Omega_N := \bigcap_{m \in \mathcal{M}_N^+} \Omega_m \quad \text{and} \quad \Xi_N := \{\mathcal{M}_N \subset \widehat{\mathcal{M}}_N \subset \mathcal{M}_N^+\},$$

where

$$\mathcal{M}_N^+ := \left\{ m \in \{1, \dots, N_T\} : [\mathfrak{c}_\varphi^2 m (\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^2 \leq 4\mathfrak{d}_T(p) \frac{NT}{\log(NT)} \right\}.$$

Moreover, recall that

$$\mathcal{M}_N = \left\{ m \in \{1, \dots, N_T\} : [\mathfrak{c}_\varphi^2 m (\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^2 \leq \frac{\mathfrak{d}_T(p)}{4} \cdot \frac{NT}{\log(NT)} \right\} \subset \mathcal{M}_N^+$$

and

$$\widehat{\mathcal{M}}_N = \left\{ m \in \{1, \dots, N_T\} : [\mathfrak{c}_\varphi^2 m (\|\widehat{\Psi}_m^{-1}\|_{\text{op}} \vee 1)]^2 \leq \mathfrak{d}_T(p) \frac{NT}{\log(NT)} \right\}.$$

The proof of Theorem 4.3 relies on the two following lemmas.

Lemma 6.3. *Under Assumptions 3.1, 3.4 and 4.1, there exists a deterministic constant $\mathfrak{c}_{6.3} > 0$, not depending on m and N , such that*

$$\mathbb{P}(\Xi_N^c) \leq \frac{\mathfrak{c}_{6.3}}{N^5}.$$

Lemma 6.4. *Consider the empirical process*

$$\nu_N(\tau) := \frac{1}{NT} \sum_{i=1}^N \int_0^t \sigma(X_s^i) \tau(X_s^i) dB_s^i; \quad \tau \in \mathcal{S}_1 \cup \dots \cup \mathcal{S}_{N_T}.$$

Under Assumptions 3.1 and 4.2, there exists a deterministic constant $\mathfrak{c}_{6.4} > 0$, not depending on N , such that for every $m \in \mathcal{M}_N$,

$$\mathbb{E} \left[\left(\left[\sup_{\tau \in \mathcal{B}_{\widehat{m}, m}} |\nu_N(\tau)| \right]^2 - p(\widehat{m}, m) \right)_+ \mathbf{1}_{\Xi_N \cap \Omega_N} \right] \leq \frac{\mathfrak{c}_{6.4}}{NT}$$

where, for every $m' \in \mathcal{M}_N$,

$$\mathcal{B}_{m, m'} := \{\tau \in \mathcal{S}_{m \vee m'} : \|\tau\|_{f_T} = 1\} \quad \text{and} \quad p(m, m') := \frac{\mathfrak{c}_{\text{cal}}}{8} \cdot \frac{m \vee m'}{NT} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i, k}| \right).$$

6.3.1. *Steps of the proof.* First of all,

$$(13) \quad \begin{aligned} \|\widehat{b}_{\widehat{m}} - b_I\|_N^2 &= \|\widehat{b}_{\widehat{m}} - b_I\|_N^2 \mathbf{1}_{\Xi_N^c} + \|\widehat{b}_{\widehat{m}} - b_I\|_N^2 \mathbf{1}_{\Xi_N} \\ &=: U_1 + U_2. \end{aligned}$$

Let us find suitable bounds on $\mathbb{E}(U_1)$ and $\mathbb{E}(U_2)$.

- **Bound on $\mathbb{E}(U_1)$.** By the definition of $\widehat{\mathcal{M}}_N$ and by Lemma 6.1,

$$\begin{aligned} \mathbb{E}(\|\widehat{b}_{\widehat{m}} - \Pi_{N, \widehat{m}}(b_I)\|_N^2 \mathbf{1}_{\Xi_N^c}) &= \mathbb{E}[(\widehat{\mathbf{E}}_m^* \widehat{\Psi}_m^{-1} \widehat{\mathbf{E}}_m) \mathbf{1}_{\Xi_N^c}] \\ &\leq \mathbb{E}(\|\widehat{\Psi}_m^{-1}\|_{\text{op}} |\widehat{\mathbf{E}}_{NT}^* \widehat{\mathbf{E}}_{NT}| \mathbf{1}_{\Xi_N^c}) \\ &\leq \sqrt{\mathfrak{d}_T(p) \frac{NT}{\log(NT)}} \mathbb{E}(|\widehat{\mathbf{E}}_{NT}^* \widehat{\mathbf{E}}_{NT}|^2)^{1/2} \mathbb{P}(\Xi_N^c)^{1/2} \\ &\leq \frac{\mathbf{c}_1 N}{\log(NT)} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}|\right) \mathbb{P}(\Xi_N^c)^{1/2} \end{aligned}$$

where $\mathbf{c}_1 > 0$ is a deterministic constant not depending on N . Then,

$$\begin{aligned} \mathbb{E}(U_1) &\leq \mathbb{E}(\|\widehat{b}_{\widehat{m}} - \Pi_{N, \widehat{m}}(b_I)\|_N^2 \mathbf{1}_{\Xi_N^c}) + \mathbb{E}(\|b_I\|_N^2 \mathbf{1}_{\Xi_N^c}) \\ &\leq \frac{\mathbf{c}_1 N}{\log(NT)} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}|\right) \mathbb{P}(\Xi_N^c)^{1/2} + \mathbf{c}_2 \mathbb{P}(\Xi_N^c)^{1/2} \end{aligned}$$

with

$$\mathbf{c}_2 = \left(\int_{-\infty}^{\infty} b_I(x)^4 f_T(x) dx \right)^{1/2}.$$

So, by Lemma 6.2, there exists a deterministic constant $\mathbf{c}_3 > 0$, not depending on N , such that

$$\mathbb{E}(U_1) \leq \frac{\mathbf{c}_3}{N} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}|\right).$$

- **Bound on $\mathbb{E}(U_2)$.** Note that

$$\begin{aligned} U_2 &= \|\widehat{b}_{\widehat{m}} - b_I\|_N^2 \mathbf{1}_{\Xi_N \cap \Omega_N^c} + \|\widehat{b}_{\widehat{m}} - b_I\|_N^2 \mathbf{1}_{\Xi_N \cap \Omega_N} \\ &=: U_{2,1} + U_{2,2}. \end{aligned}$$

On the one hand, by Lemma 6.2, there exists a deterministic constant $\mathbf{c}_4 > 0$, not depending on N , such that

$$\mathbb{P}(\Xi_N \cap \Omega_N^c) \leq \sum_{m \in \mathcal{M}_N^+} \mathbb{P}(\Omega_m^c) \leq \frac{\mathbf{c}_4}{N^5}.$$

Then, as for $\mathbb{E}(U_1)$, there exists a deterministic constant $\mathbf{c}_5 > 0$, not depending on N , such that

$$\mathbb{E}(U_{2,1}) \leq \frac{\mathbf{c}_5}{N} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}|\right).$$

On the other hand,

$$\gamma_N(\tau') - \gamma_N(\tau) = \|\tau' - b\|_N^2 - \|\tau - b\|_N^2 - 2\nu_N(\tau' - \tau)$$

for every $\tau, \tau' \in \mathcal{S}_1 \cup \dots \cup \mathcal{S}_{N_T}$. Moreover, since

$$\widehat{m} = \arg \min_{m \in \widehat{\mathcal{M}}_N} \{-\|\widehat{b}_m\|_N^2 + \text{pen}(m)\} = \arg \min_{m \in \widehat{\mathcal{M}}_N} \{\gamma_N(\widehat{b}_m) + \text{pen}(m)\},$$

for every $m \in \widehat{\mathcal{M}}_N$,

$$(14) \quad \gamma_N(\widehat{b}_{\widehat{m}}) + \text{pen}(\widehat{m}) \leq \gamma_N(\widehat{b}_m) + \text{pen}(m).$$

On the event $\Xi_N = \{\mathcal{M}_N \subset \widehat{\mathcal{M}}_N \subset \mathcal{M}_N^+\}$, Inequality (14) remains true for every $m \in \mathcal{M}_N$. Then, on Ξ_N , for any $m \in \mathcal{M}_N$, since $\mathcal{S}_m + \mathcal{S}_{\widehat{m}} \subset \mathcal{S}_{m \vee \widehat{m}}$ under Assumption 4.1,

$$\begin{aligned} \|\widehat{b}_{\widehat{m}} - b_I\|_N^2 &\leq \|\widehat{b}_m - b_I\|_N^2 + 2\nu_N(\widehat{b}_{\widehat{m}} - \widehat{b}_m) + \text{pen}(m) - \text{pen}(\widehat{m}) \\ &\leq \|\widehat{b}_m - b_I\|_N^2 + \frac{1}{8}\|\widehat{b}_{\widehat{m}} - \widehat{b}_m\|_{f_T}^2 \\ &\quad + 8 \left(\left[\sup_{\tau \in \mathcal{B}_{m, \widehat{m}}} |\nu_N(\tau)| \right]^2 - p(m, \widehat{m}) \right) + \text{pen}(m) + 8p(m, \widehat{m}) - \text{pen}(\widehat{m}). \end{aligned}$$

Since $\|\cdot\|_{f_T}^2 \mathbf{1}_{\Omega_N} \leq 2\|\cdot\|_N^2 \mathbf{1}_{\Omega_N}$ on $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_{\max(\mathcal{M}_N^+)}$, and since $8p(m, \widehat{m}) \leq \text{pen}(m) + \text{pen}(\widehat{m})$, on $\Xi_N \cap \Omega_N$,

$$\|\widehat{b}_{\widehat{m}} - b_I\|_N^2 \leq 3\|\widehat{b}_m - b_I\|_N^2 + 4\text{pen}(m) + 16 \left(\left[\sup_{\tau \in \mathcal{B}_{m, \widehat{m}}} |\nu_N(\tau)| \right]^2 - p(m, \widehat{m}) \right).$$

So, by Lemma 6.3,

$$\begin{aligned} \mathbb{E}(U_{2,2}) &\leq \min_{m \in \mathcal{M}_N} \{ \mathbb{E}(3\|\widehat{b}_m - b_I\|_N^2 \mathbf{1}_{\Xi_N}) + 4\text{pen}(m) \} + \frac{16\mathbf{c}_{6.4}}{NT} \\ &\leq \mathbf{c}_6 \min_{m \in \mathcal{M}_N} \left\{ \inf_{\tau \in \mathcal{S}_m} \|\tau - b_I\|_{f_T}^2 + \frac{m}{NT} \left(1 + \frac{1}{N} \sum_{i \neq k} |R_{i,k}| \right) \right\} + \frac{\mathbf{c}_6}{N} \end{aligned}$$

where $\mathbf{c}_6 > 0$ is a deterministic constant not depending on N .

6.3.2. *Proof of Lemma 6.3.* Note that

$$\Xi_N^c = \{\mathcal{M}_N \not\subset \widehat{\mathcal{M}}_N\} \cup \{\widehat{\mathcal{M}}_N \not\subset \mathcal{M}_N^+\}.$$

The proof of Lemma 6.3 is dissected in three steps. Step 1 deals with a bound on $\mathbb{P}(\mathcal{M}_N \not\subset \widehat{\mathcal{M}}_N)$, step 2 with a bound on $\mathbb{P}(\|\widehat{\Psi}_m - \Psi_m\|_{\text{op}} > \delta)$, $\delta > 0$, and step 3 with a bound on $\mathbb{P}(\widehat{\mathcal{M}}_N \not\subset \mathcal{M}_N^+)$.

Step 1. On $\{\mathcal{M}_N \not\subset \widehat{\mathcal{M}}_N\} = \{\max(\mathcal{M}_N) > \max(\widehat{\mathcal{M}}_N)\}$, there exists $m \in \{1, \dots, N_T\}$ such that

$$[\mathbf{c}_\varphi^2 m (\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^2 \leq \frac{\mathfrak{d}_T(p)}{4} \cdot \frac{NT}{\log(NT)} \quad \text{and} \quad [\mathbf{c}_\varphi^2 m (\|\widehat{\Psi}_m^{-1}\|_{\text{op}} \vee 1)]^2 > \mathfrak{d}_T(p) \frac{NT}{\log(NT)}.$$

The first inequality is equivalent to

$$\mathbf{c}_\varphi^4 m^2 \|\Psi_m^{-1}\|_{\text{op}}^2 \leq \frac{\mathfrak{d}_T(p)}{4} \cdot \frac{NT}{\log(NT)} \quad \text{and} \quad \mathbf{c}_\varphi^4 m^2 \leq \frac{\mathfrak{d}_T(p)}{4} \cdot \frac{NT}{\log(NT)},$$

and then the second one leads to

$$\begin{aligned} \mathfrak{d}_T(p) \frac{NT}{\log(NT)} &< \mathbf{c}_\varphi^4 m^2 \|\widehat{\Psi}_m^{-1}\|_{\text{op}}^2 \leq 2\mathbf{c}_\varphi^4 m^2 (\|\widehat{\Psi}_m^{-1} - \Psi_m^{-1}\|_{\text{op}}^2 + \|\Psi_m^{-1}\|_{\text{op}}^2) \\ &\leq 2\mathbf{c}_\varphi^4 m^2 \|\widehat{\Psi}_m^{-1} - \Psi_m^{-1}\|_{\text{op}}^2 + \frac{\mathfrak{d}_T(p)}{2} \cdot \frac{NT}{\log(NT)}. \end{aligned}$$

So,

$$\begin{aligned} \{\mathcal{M}_N \not\subset \widehat{\mathcal{M}}_N\} &\subset \bigcup_{m \in \mathcal{M}_N} \left\{ \frac{\mathfrak{d}_T(p)}{4} \cdot \frac{NT}{\log(NT)} \leq \mathbf{c}_\varphi^4 m^2 \|\widehat{\Psi}_m^{-1} - \Psi_m^{-1}\|_{\text{op}}^2 \right\} \\ &\subset \bigcup_{m \in \mathcal{M}_N} \{ \|\Psi_m^{-1}\|_{\text{op}} < \|\widehat{\Psi}_m^{-1} - \Psi_m^{-1}\|_{\text{op}} \} \subset \bigcup_{m \in \mathcal{M}_N} \Omega_m^c \end{aligned}$$

and, since $\mathfrak{d}_T(p)/4 \leq \mathbf{c}_T(p)/2$, by Lemma 6.2,

$$\mathbb{P}(\mathcal{M}_N \not\subset \widehat{\mathcal{M}}_N) \leq \sum_{m \in \mathcal{M}_N} \mathbb{P}(\Omega_m^c) \leq \frac{\mathbf{c}_1}{N^5}$$

where $\mathbf{c}_1 > 0$ is a deterministic constant not depending on N .

Step 2. First of all, note that

$$\|\widehat{\Psi}_m - \Psi_m\|_{\text{op}} \leq M_N + R_N,$$

where

$$M_N := \frac{1}{N} \left\| \sum_{i=1}^N (\widehat{\Psi}_m(X^i) - \mathbb{E}_{i-1}(\widehat{\Psi}_m(X^i))) \right\|_{\text{op}} \quad \text{and} \quad R_N := \frac{1}{N} \left\| \sum_{i=1}^N (\mathbb{E}_{i-1}(\widehat{\Psi}_m(X^i)) - \mathbb{E}(\widehat{\Psi}_m(X^i))) \right\|_{\text{op}}.$$

On the one hand, for any $\delta > 0$, let us establish a suitable bound on $\mathbb{P}(M_N > \delta)$. For every $i \in \{1, \dots, N\}$, since

$$\widetilde{\Psi}_m(X^i) := \frac{1}{N} (\widehat{\Psi}_m(X^i) - \mathbb{E}_{i-1}(\widehat{\Psi}_m(X^i)))$$

is a symmetric matrix, by Jensen's inequality and by Inequality (11),

$$\begin{aligned} \widetilde{\Psi}_m(X^i)^2 &\preceq \lambda_{\max}[\widetilde{\Psi}_m(X^i)^2] \mathbf{I} = \frac{1}{N^2} \|\widehat{\Psi}_m(X^i) - \mathbb{E}_{i-1}(\widehat{\Psi}_m(X^i))\|_{\text{op}}^2 \mathbf{I} \\ &\preceq \frac{2}{N^2} (\|\widehat{\Psi}_m(X^i)\|_{\text{op}}^2 + \mathbb{E}_{i-1}(\|\widehat{\Psi}_m(X^i)\|_{\text{op}}^2)) \mathbf{I} \preceq \mathbf{A}_i^2 \end{aligned}$$

with

$$\mathbf{A}_i^2 := \frac{4L(m)^2}{N^2} \mathbf{I}.$$

So, by Azuma's inequality for matrix martingales (see Tropp [22], Theorem 7.1),

$$\mathbb{P}(M_N > \delta) = \mathbb{P} \left(\left\| \sum_{i=1}^N \widetilde{\Psi}_m(X^i) \right\|_{\text{op}} > \delta \right) \leq m \exp \left(-\frac{\delta^2 N}{32L(m)^2} \right).$$

On the other hand, let us establish a suitable bound on $\mathbb{P}(R_N > \delta)$. By the definition of \mathcal{I}_N ,

$$R_N = \frac{1}{N} \left\| \sum_{i \in \mathcal{I}_N} (\mathbb{E}_{i-1}(\widehat{\Psi}_m(X^i)) - \mathbb{E}(\widehat{\Psi}_m(X^i))) \right\|_{\text{op}}.$$

Then, by Markov's inequality and Jensen's inequality (usual and conditional),

$$\begin{aligned} \mathbb{P}(R_N > \delta) &\leq \frac{\mathbb{E}(R_N^p)}{\delta^p} \leq \frac{|\mathcal{I}_N|^{p-1}}{\delta^p N^p} \sum_{i \in \mathcal{I}_N} \mathbb{E}[\|\mathbb{E}_{i-1}[\widehat{\Psi}_m(X^i) - \mathbb{E}(\widehat{\Psi}_m(X^i))]\|_{\text{op}}^p] \\ &\leq \frac{|\mathcal{I}_N|^p}{\delta^p N^p} \mathbb{E}(\|\widehat{\Psi}_m(X) - \mathbb{E}(\widehat{\Psi}_m(X))\|_{\text{op}}^p) \leq \frac{2^p |\mathcal{I}_N|^p}{\delta^p N^p} L(m)^p. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}(\|\widehat{\Psi}_m - \Psi_m\|_{\text{op}} > \delta) &\leq \mathbb{P} \left(\left\{ M_N > \frac{\delta}{2} \right\} \cup \left\{ R_N > \frac{\delta}{2} \right\} \right) \\ &\leq m \exp \left(-\frac{\delta^2 N}{128L(m)^2} \right) + \frac{2^{2p} |\mathcal{I}_N|^p}{\delta^p N^p} L(m)^p. \end{aligned}$$

Step 3. On $\{\widehat{\mathcal{M}}_N \not\subset \mathcal{M}_N^+\} = \{\max(\widehat{\mathcal{M}}_N) > \max(\mathcal{M}_N^+)\}$, there exists $m \in \{1, \dots, N_T\}$ such that

$$[\mathbf{c}_\varphi^2 m (\|\widehat{\Psi}_m^{-1}\|_{\text{op}} \vee 1)]^2 \leq \mathfrak{d}_T(p) \frac{NT}{\log(NT)} \quad \text{and} \quad [\mathbf{c}_\varphi^2 m (\|\Psi_m^{-1}\|_{\text{op}} \vee 1)]^2 > 4\mathfrak{d}_T(p) \frac{NT}{\log(NT)}.$$

The first inequality is equivalent to

$$\mathbf{c}_\varphi^4 m^2 \|\widehat{\Psi}_m^{-1}\|_{\text{op}}^2 \leq \mathfrak{d}_T(p) \frac{NT}{\log(NT)} \quad \text{and} \quad \mathbf{c}_\varphi^4 m^2 \leq \mathfrak{d}_T(p) \frac{NT}{\log(NT)},$$

and then the second one leads to

$$\begin{aligned} 4\mathfrak{d}_T(p) \frac{NT}{\log(NT)} &< \mathfrak{c}_\varphi^4 m^2 \|\Psi_m^{-1}\|_{\text{op}}^2 \leq 2\mathfrak{c}_\varphi^4 m^2 (\|\Psi_m^{-1} - \widehat{\Psi}_m^{-1}\|_{\text{op}}^2 + \|\widehat{\Psi}_m^{-1}\|_{\text{op}}^2) \\ &\leq 2\mathfrak{c}_\varphi^4 m^2 \|\Psi_m^{-1} - \widehat{\Psi}_m^{-1}\|_{\text{op}}^2 + 2\mathfrak{d}_T(p) \frac{NT}{\log(NT)}. \end{aligned}$$

Moreover, for every $m \in \{1, \dots, N_T\}$,

$$\begin{aligned} &\{\|\Psi_m^{-1} - \widehat{\Psi}_m^{-1}\|_{\text{op}} > \|\widehat{\Psi}_m^{-1}\|_{\text{op}}\} \\ &\subset \left\{ \|\widehat{\Psi}_m^{-1/2} \Psi_m \widehat{\Psi}_m^{-1/2} - \mathbf{I}\|_{\text{op}} > \frac{1}{2} \right\} \subset \left\{ \|\widehat{\Psi}_m - \Psi_m\|_{\text{op}} > \frac{1}{2} \|\widehat{\Psi}_m^{-1}\|_{\text{op}}^{-1} \right\} \end{aligned}$$

by interchanging $\widehat{\Psi}_m$ and Ψ_m in the proof of Comte and Genon-Catalot [6], Proposition 4.(ii). So,

$$\begin{aligned} \{\mathcal{M}_N \not\subset \widehat{\mathcal{M}}_N\} &\subset \bigcup_{\mathfrak{c}_\varphi^4 m^2 \leq \mathfrak{d}_T(p) NT / \log(NT)} \left\{ 2\mathfrak{c}_\varphi^4 m^2 \|\widehat{\Psi}_m^{-1}\|_{\text{op}}^2 < 2\mathfrak{d}_T(p) \frac{NT}{\log(NT)} \leq 2\mathfrak{c}_\varphi^4 m^2 \|\Psi_m^{-1} - \widehat{\Psi}_m^{-1}\|_{\text{op}}^2 \right\} \\ &\subset \bigcup_{\mathfrak{c}_\varphi^4 m^2 \leq \mathfrak{d}_T(p) NT / \log(NT)} \left\{ \|\widehat{\Psi}_m - \Psi_m\|_{\text{op}} > \frac{m}{2} \sqrt{\frac{\log(NT)}{\mathfrak{d}_T(p) NT}} \right\} \end{aligned}$$

and, by the previous step, Assumptions 3.4 and 4.1, and since $p \geq 12$,

$$\begin{aligned} \mathbb{P}(\mathcal{M}_N \not\subset \widehat{\mathcal{M}}_N) &\leq \sum_{\mathfrak{c}_\varphi^4 m^2 \leq \mathfrak{d}_T(p) NT / \log(NT)} \left(m \exp\left(-\frac{N}{512L(m)^2} \cdot \frac{m^2 \log(NT)}{\mathfrak{d}_T(p) NT}\right) \right. \\ &\quad \left. + \frac{2^{3p} |\mathcal{I}_N|^p}{N^p} \cdot \frac{\mathfrak{d}_T(p)^{p/2} (NT)^{p/2}}{m^p \log(NT)^{p/2}} L(m)^p \right) \\ &\leq \sum_{\mathfrak{c}_\varphi^4 m^2 \leq \mathfrak{d}_T(p) NT / \log(NT)} \left(m \exp\left(-\frac{1}{512\mathfrak{c}_\varphi^4} \cdot \frac{\log(NT)}{\mathfrak{d}_T(p) T}\right) \right. \\ &\quad \left. + \frac{2^{3p} |\mathcal{I}_N|^p}{N^{p/2}} \cdot \frac{\mathfrak{d}_T(p)^{p/2} T^{p/2}}{\log(NT)^{p/2}} \mathfrak{c}_\varphi^{2p} \right) \\ &\leq \sum_{\mathfrak{c}_\varphi^4 m^2 \leq \mathfrak{d}_T(p) NT / \log(NT)} \left(\frac{m}{N^{1+p/2}} + \frac{2^{3p} \mathfrak{c}_\varphi^{2p} \mathfrak{c}_{3.4}(p)}{N^6} \mathfrak{d}_T(p)^{p/2} T^{p/2} \right) \leq \frac{\mathfrak{c}_2}{N^5} \end{aligned}$$

where $\mathfrak{c}_2 > 0$ is a deterministic constant not depending on N .

6.3.3. *Proof of Lemma 6.4.* The proof of Lemma 6.4 is dissected in two steps.

Step 1. Consider $\tau \in \mathcal{S}_1 \cup \dots \cup \mathcal{S}_{N_T}$ and the martingale $(M_N(\tau)_t)_{t \in [0, T]}$ defined by

$$M_N(\tau)_t := \sum_{i=1}^N \int_0^t \sigma(X_s^i) \tau(X_s^i) dB_s^i; \quad \forall t \in [0, T].$$

Note that $\nu_N(\tau) = M_N(\tau)_T / (NT)$. Since $d\langle B^i, B^k \rangle_t = R_{i,k} dt$ for every $i, k \in \{1, \dots, N\}$,

$$\begin{aligned} \langle M_N(\tau) \rangle_T &= \sum_{i,k=1}^N R_{i,k} \int_0^T \sigma(X_t^i) \sigma(X_t^k) \tau(X_t^i) \tau(X_t^k) dt \\ &= \int_0^T (\sigma(X_t^i) \tau(X_t^i))_i^* \times R \times (\sigma(X_t^i) \tau(X_t^i))_i dt \\ &\leq \|R\|_{\text{op}} \int_0^T \|(\sigma(X_t^i) \tau(X_t^i))_{1 \leq i \leq N}\|_{2,N}^2 dt \\ &\leq \|R\|_{\text{op}} \|\sigma\|_{\infty}^2 \int_0^T \left(\sum_{i=1}^N \tau(X_t^i)^2 \right) dt = NT \|R\|_{\text{op}} \|\sigma\|_{\infty}^2 \|\tau\|_N^2. \end{aligned}$$

Then, by Assumption 4.2 and Bernstein's inequality for local martingales (see Revuz and Yor [21], p. 153), for any $\varepsilon, \nu > 0$,

$$\begin{aligned} \mathbb{P}(\nu_N(\tau) \geq \varepsilon, \|\tau\|_N^2 \leq \nu^2) &\leq \mathbb{P}(M_N(\tau)_T^* \geq NT\varepsilon, \langle M_N(\tau) \rangle_T \leq NT\nu^2 \|R\|_{\text{op}} \|\sigma\|_{\infty}^2) \\ &\leq \exp\left(-\frac{NT\varepsilon^2}{2\nu^2 \|\sigma\|_{\infty}^2 \mathfrak{r}}\right) \quad \text{with } \mathfrak{r} = 1 + \mathfrak{m}_{4.2}. \end{aligned}$$

Step 2. By using the bound of step 1 and by following the pattern of the proof of Baraud *et al.* [2], Proposition 6.1, the purpose of this step is to find a suitable bound on

$$\mathbb{E} \left[\left(\left[\sup_{\tau \in \mathcal{B}_{m,m'}} |\nu_N(\tau)| \right]^2 - p(m, m') \right)_+ \mathbf{1}_{\Xi_N \cap \Omega_N} \right]; \quad m, m' \in \mathcal{M}_N.$$

Consider $\delta_0 \in (0, 1)$ and let $(\delta_n)_{n \in \mathbb{N}^*}$ be the real sequence defined by

$$\delta_n := \delta_0 2^{-n}; \quad \forall n \in \mathbb{N}^*.$$

Since $\mathcal{S}_{m \vee m'}$ is a vector subspace of $\mathbb{L}^2([0, T], f_T(x) dx)$ of dimension $m \vee m'$, for any $n \in \mathbb{N}$, there exists $T_n \subset \mathcal{B}_{m,m'}$ such that $|T_n| \leq (3/\delta_n)^{m \vee m'}$ and, for any $\tau \in \mathcal{B}_{m,m'}$,

$$\exists f_n \in T_n : \|\tau - f_n\|_{f_T} \leq \delta_n.$$

In particular, note that

$$\tau = f_0 + \sum_{n=1}^{\infty} (f_n - f_{n-1}).$$

Then, for any sequence $(\Delta_n)_{n \in \mathbb{N}}$ of elements of $(0, \infty)$ such that $\Delta = \sum_{n \in \mathbb{N}} \Delta_n < \infty$,

$$\begin{aligned} &\left\{ \left[\sup_{\tau \in \mathcal{B}_{m,m'}} \nu_N(\tau) \right]^2 > \Delta^2 \right\} \\ &= \left\{ \exists (f_n)_{n \in \mathbb{N}} \in \prod_{n=0}^{\infty} T_n : \nu_N(f_0) + \sum_{n=1}^{\infty} \nu_N(f_n - f_{n-1}) > \Delta \right\} \\ &\subset \left\{ \exists (f_n)_{n \in \mathbb{N}} \in \prod_{n=0}^{\infty} T_n : \nu_N(f_0) > \Delta_0 \text{ or } [\exists n \in \mathbb{N}^* : \nu_N(f_n - f_{n-1}) > \Delta_n] \right\} \\ &\subset \bigcup_{f_0 \in T_0} \{ \nu_N(f_0) > \Delta_0 \} \cup \bigcup_{n=1}^{\infty} \bigcup_{(f_{n-1}, f_n) \in \mathbb{T}_n} \{ \nu_N(f_n - f_{n-1}) > \Delta_n \} \end{aligned}$$

with $\mathbb{T}_n = T_{n-1} \times T_n$ for every $n \in \mathbb{N}^*$. Moreover, $\|f_0\|_{f_T}^2 \leq \delta_0^2$,

$$\|f_n - f_{n-1}\|_{f_T}^2 \leq 2\delta_{n-1}^2 + 2\delta_n^2 = \frac{5}{2}\delta_{n-1}^2; \quad \forall n \in \mathbb{N}^*,$$

and $\|\cdot\|_N^2 \mathbf{1}_{\Omega_N} \leq 3/2 \|\cdot\|_{f_T}^2 \mathbf{1}_{\Omega_N}$ on $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_{\max(\mathcal{M}_N^+)}$. So, by step 1,

$$\begin{aligned}
& \mathbb{P} \left(\left\{ \left[\sup_{\tau \in \mathcal{B}_{m,m'}} \nu_N(\tau) \right]^2 > \Delta^2 \right\} \cap \Xi_N \cap \Omega_N \right) \\
& \leq \sum_{f_0 \in \mathcal{T}_0} \exp \left(-\frac{NT\Delta_0^2}{3\|f_0\|_{f_T}^2 \|\sigma\|_\infty^2 \mathfrak{r}} \right) \\
& \quad + \sum_{n=1}^{\infty} \sum_{(f_{n-1}, f_n) \in \mathcal{T}_n} \exp \left(-\frac{NT\Delta_n^2}{3\|f_n - f_{n-1}\|_{f_T}^2 \|\sigma\|_\infty^2 \mathfrak{r}} \right) \\
(15) \quad & \leq \exp \left(h_0 - \frac{NT\Delta_0^2}{3\delta_0^2 \|\sigma\|_\infty^2 \mathfrak{r}} \right) + \sum_{n=1}^{\infty} \exp \left(h_{n-1} + h_n - \frac{NT\Delta_n^2}{15/2\delta_{n-1}^2 \|\sigma\|_\infty^2 \mathfrak{r}} \right)
\end{aligned}$$

with $h_n = \log(|T_n|)$ for every $n \in \mathbb{N}$. Now, let us take Δ_0 such that

$$h_0 - \frac{NT\Delta_0^2}{3\delta_0^2 \|\sigma\|_\infty^2 \mathfrak{r}} = -(m \vee m' + x) \quad \text{with } x > 0,$$

which leads to

$$\Delta_0 = \left[\frac{3\delta_0^2 \|\sigma\|_\infty^2 \mathfrak{r}}{NT} (m \vee m' + x + h_0) \right]^{1/2},$$

and for every $n \in \mathbb{N}^*$, let us take Δ_n such that

$$h_{n-1} + h_n - \frac{NT\Delta_n^2}{15/2\delta_{n-1}^2 \|\sigma\|_\infty^2 \mathfrak{r}} = -(m \vee m' + x + n),$$

which leads to

$$\Delta_n = \left[\frac{15/2\delta_{n-1}^2 \|\sigma\|_\infty^2 \mathfrak{r}}{NT} (m \vee m' + x + n + h_{n-1} + h_n) \right]^{1/2}.$$

For this appropriate sequence $(\Delta_n)_{n \in \mathbb{N}}$,

$$\mathbb{P} \left(\left\{ \left[\sup_{\tau \in \mathcal{B}_{m,m'}} \nu_N(\tau) \right]^2 > \Delta^2 \right\} \cap \Xi_N \cap \Omega_N \right) \leq e^{-x} e^{-(m \vee m')} \left(1 + \sum_{n=1}^{\infty} e^{-n} \right) \leq 1.6e^{-x} e^{-(m \vee m')}$$

by Inequality (15), and

$$\begin{aligned}
\Delta^2 & \leq \frac{3\|\sigma\|_\infty^2 \mathfrak{r}}{NT} \left[\delta_0 [(m \vee m' + x)^{1/2} + h_0^{1/2}] + \sqrt{\frac{5}{2}} \sum_{n=1}^{\infty} \delta_{n-1} [(m \vee m' + x)^{1/2} + (n + h_{n-1} + h_n)^{1/2}] \right]^2 \\
& \leq \frac{3\|\sigma\|_\infty^2 \mathfrak{r}}{NT} \delta^{(1)} (m \vee m' + x) + \frac{3\|\sigma\|_\infty^2 \mathfrak{r}}{NT} \delta^{(2)} \leq \frac{3\|\sigma\|_\infty^2 \mathfrak{r}}{NT} (\delta^{(1)} + \delta^{(2)}) (m \vee m' + x)
\end{aligned}$$

with

$$\delta^{(1)} = 2 \left(\delta_0 + \sqrt{\frac{5}{2}} \sum_{n=1}^{\infty} \delta_{n-1} \right)^2 \quad \text{and} \quad \delta^{(2)} = 2 \left(\delta_0 h_0^{1/2} + \sqrt{\frac{5}{2}} \sum_{n=1}^{\infty} \delta_{n-1} (n + h_{n-1} + h_n)^{1/2} \right)^2.$$

Then,

$$\mathbb{P} \left(\left[\sup_{\tau \in \mathcal{B}_{m,m'}} \nu_N(\tau) \right]^2 - \frac{\rho}{\mathfrak{c}_{\text{cal}} R_N} p(m, m') > \frac{\rho}{NT} x \right) \leq 1.6e^{-x} e^{-(m \vee m')}$$

with

$$\rho = 3\|\sigma\|_\infty^2 \mathfrak{r} (\delta^{(1)} + \delta^{(2)}) \quad \text{and} \quad R_N = 1 + \sum_{i \neq k} |R_{i,k}|.$$

So, by taking $\mathfrak{c}_{\text{cal}} \geq \rho > \rho/R_N$ and $y = \rho x/(NT)$,

$$\mathbb{P} \left(\left[\sup_{\tau \in \mathcal{B}_{m,m'}} \nu_N(\tau) \right]^2 - p(m, m') > y \right) \leq 1.6e^{-NTy/\rho} e^{-(m \vee m')}.$$

Therefore,

$$\begin{aligned} \mathbb{E} \left[\left(\left[\sup_{\tau \in \mathcal{B}_{m,m'}} \nu_N(\tau) \right]^2 - p(m, m') \right)_+ \right] &= \int_0^\infty \mathbb{P} \left(\left[\sup_{\tau \in \mathcal{B}_{m,m'}} \nu_N(\tau) \right]^2 - p(m, m') > y \right) dy \\ &\leq 1.6\rho \frac{e^{-(m \vee m')}}{NT}. \end{aligned}$$

A union-bound allows to conclude.

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