Boundary density and Voronoi approximation of irregular sets

Raphaël Lachiéze-Rey* and Sergio Vega †

December 12, 2014

Contents

In this paper, we study the inner and outer boundary densities of some sets with self-similar boundary. These quantities show up to be crucial for the Voronoi approximation of the set.

Notations In all the following, $d(.,.)$ designates the Euclidean distance between points or subsets of $\mathbb{R}^d$, $\mathrm{cl}(E)$ the closure of a set $E \subset \mathbb{R}^d$, $\mathrm{int}(E)$ its interior, $\mathrm{diam}(E)$ its diameter, Vol is the d-dimensional Lebesgue measure and κ_d is the volume of the Euclidean unit ball. $B(x,r)$ is the open ball with center $x \in \mathbb{R}^d$ and radius $r \geqslant 0$.

*raphael.lachieze-rey@parisdescartes.fr, Laboratoire MAP5 (UMR CNRS 8145), Université Paris Descartes, Sorbonne Paris Cité

†Laboratoire MAP5 (UMR CNRS 8145), Université Paris Descartes, Sorbonne Paris Cité

1 Introduction and main result

Let $K \subset \mathbb{R}^d$ be a measurable set. For $x \in \mathbb{R}^d$, we define the density of K at radius r in x by

$$f_r(x) = f_r^K(x) = \frac{\mathrm{Vol}(K \cap B(x,r))}{\mathrm{Vol}(B(x,r))}.$$

We define the *lower density* of K in x by $\liminf_r f_r$, the *upper density* by $\limsup_r f_r(x)$, and call it the density if those two values coincide. We call ∂K the topological boundary of K. In Geometric measure theory, the points with density 0 or 1 are considered resp. as the measure-theoretic exterior and interior of K, and other points constitute $\partial^* K$ the *essential boundary* of K. Determining that most points of the boundary of K have a positive lower density is a good assessment of K's regularity. Federer [1] proved that if K is a measurable set with finite measure-theoretic perimeter then most of the essential boundary points have density $1/2$.

Theorem 1. *[1, Th.3.60] Let $K \subset \mathbb{R}^d$ be a measurable set. Assume that K has finite perimeter in the sense of [1, Definition 3.35]. Call $\mathcal{H}_{d-1}$ the d-dimensional Hausdorff measure. Then for $\mathcal{H}_{d-1}$-a.e. point of $\partial^* K$, we have*

$$\lim_r f_r(x) = \frac{1}{2}.$$

The result extends to $\mathcal{H}_{d-1}$-a.e. point of ∂K if K has Lipschitz boundary because in this case $\partial K \subset \partial^* K$ (end of [1, Section 3.5]) (some sets K with finite perimeter have a boundary ∂K with positive Lebesgue measure). Back to general K with finite perimeter, there is furthermore the existence of a tangent plane in each point with density $1/2$.

In this context, $\mathcal{H}_{d-1}(\partial^* K) > 0$, and $\mathcal{H}_{d-1} - a.e.$ indeed corresponds to "most" of the points. If ∂K is irregular, do most of the points still have a positive density? In this paper we study the boundary density of sets which have a boundary with self-similar properties. In general, such boundaries have a Hausdorff dimension $s > d-1$ and don't have finite perimeter. We prove that under sufficient assumptions, for such a self-similar boundary, $\liminf f_r^K > 0$ on all of the boundary, or on most of it. Remark that if this assertion holds for K, it does not automatically hold for K^c, even though they share the same boundary. We actually give conditions ensuring that both $\liminf f_r^K > 0$ and $\liminf f_r^{K^c} > 0$.

Such a regularity assumption is also related to the *rolling ball* or *sliding ball* conditions that one may find in integral geometry, or other domains. The rolling ball condition can take several formulations. For some compact set K the requirement is that one can slide a ball of radius r in the set ∂K_{Cr}^- formed by points of K at distance less than Cr of K^c (for some constant $C > 0$), or more formally that for every point $x \in \partial K, r > 0$ there is a ball of radius r contained in $K \cap B(x, Cr)$. Such an assertion automatically implies that the density at avery point of the boundary is larger or equal to C^{-d}, and therefore positive.

Our motivations come from the problem of Voronoi approximation, concerned with the quality of the approximation of K by a polygonal set based on a finite sample of K. Roughly speaking, one draws a random finite set of points χ in the region where K is supposed to lie, and approximates K by the set $K_\chi \subset \mathbb{R}^d$ formed by all points that are closer to a point from $\chi \cap K$ than from any point of $\chi \cap K^c$. It is not clear whether using a random or deterministic set of points gives a better approximation, but the answer to this question clearly depends on fine positioning of K, which is itself unknown, and therefore uncertainty is an intrinsic part of the problem. We consider here the case of a Poisson process, or a large number n of IID points uniformly distributed in the region where K lies. It is suggested in [6] that measuring the volumes of the Voronoi cells as it is done here is more efficient than simply counting the proportion of points falling into K.

This set approximation might serve in reconstruction in geostatistics, or in image analysis, it has first been introduced by Einmahl and Khmaladze [3] as a discriminating statistic in the two-sample problem. These authors proved a strong law of large numbers in dimension 1 for the volume approximation $\varphi(\chi) = \mathrm{Vol}(K^\chi)$. Reitzner and Heveling [6] proved that if K is convex and compact and $\chi = \chi_\lambda$ is a homogeneous Poisson process with intensity λ, $\mathbf{E}\varphi(\chi) = \mathrm{Vol}(K)$, and $\mathbf{Var}(\varphi(\chi)) \leq c\lambda^{-1-1/d}S(K)$ where c is an explicit constant and $S(K)$ is the surface area of K. They also established that $\mathbf{E}\varphi_{\mathrm{Per}}(\chi) = c'\lambda^{-1/d}S(K)(1+O(\lambda^{-1/d}))$ and $\mathbf{Var}(\varphi_{\mathrm{Per}}(\chi)) \leq c'\lambda^{-1-1/d}S(K)$, where $\varphi_{\mathrm{Per}} = \mathrm{Vol}(K\Delta K^\chi)$ aims at estimating the perimeter of the set. Reitzner, Spodarev and Zaporozhets [10] extended these results to sets with finite variational perimeter, and also gave upper bounds for $\mathbf{E}|\varphi(\chi_\lambda)^q - \mathrm{Vol}(K)^q|$ for $q \geq 1$. Schulte proved a similar lower bound for the variance, i.e. $CS(K)\lambda^{-1-1/d} \leq \mathbf{Var}(\varphi(\chi))$ with K a convex body and C a universal constant, and the corresponding CLT

$$d_W\left(\frac{\varphi(\chi) - \mathbf{E}\varphi(\chi)}{\sqrt{\mathbf{Var}(\varphi(\chi))}}, N\right) \underset{\lambda\to\infty}{\longrightarrow} 0.$$

For Binomial input, Penrose proved the remarkable fact that for χ_n consisting in n iid variables with density $\kappa(x) > 0$ on $[0,1]^d$,

$$\mathbf{E}\varphi(\chi_n) \to \mathrm{Vol}(K),$$

independently of any assumption on K's boundary.

This paper is motivated by the results of [7], in which it was proved that if ∂K has Minkowski dimension s, then under sufficient assumptions regarding K's boundary densities (precised in section 3.1) the following variance asymptotics hold

$$0 < \liminf_n \mathbf{Var}(\mathrm{Vol}(K_{\chi_n}))n^{s/d-2} \leqslant \limsup_n \mathbf{Var}(\mathrm{Vol}(K_{\chi_n}))n^{s/d-2} < \infty.$$

An estimation of the Kolmogorov distance between the renormalized approximation volume and the standard Gaussian law was also given. For $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that for $n \geqslant 1$

$$\sup_{t\in\mathbb{R}}\left|\mathbf{P}\left(\frac{\mathrm{Vol}(K_{\chi_n}) - \mathbf{E}\mathrm{Vol}(K_{\chi_n})}{\sqrt{\mathbf{Var}(\mathrm{Vol}(K_{\chi_n}))}} \geqslant t\right) - \mathbf{P}(N \geqslant t)\right| \leqslant C_\varepsilon n^{-s/2d+\varepsilon}$$

where N is a standard normal variable.

Our aim here is to apply these results to sets with a self-similar boundary. Intuitively, for $x \in \partial K$ and $r > 0$, due to the self-similarity, it should be possible to make geometric assumptions about $B(x,r) \cap \partial K$ that don't depend and r. The densities $f_r^K, f_r^{K^c}$ should have continuous and somehow periodical fluctuations in r, and therefore a positive infimum, which would classify the set here as *well-behaved* with respect to Voronoi approximation. This is confirmed by Theorem 2, that states that if ∂K, self-similar with dimension $s < d$, satisfies Assumption 1, then for some $\varepsilon > 0$, $f_r^K, f_r^{K^c} > \varepsilon$ on the boundary uniformly in $r > 0$. Theorem 2 applies for instance to the Von Koch flake in dimension 2.

However, some sets with self-similar boundary do not fall under the scope of this result, and we also give example of a self-similar set K_{cantor} with Cantor-like self-similar boundary not satisfying the conclusions of the theorem. Simulations we ran suggest that this irregularity of K_{cantor}'s boundary densities reflects on the behaviour of its Voronoi approximation.

It is remarkable that the densities f_r^K and $f_r^{K^c}$ are also crucial when one studies the quality of the approximation with regard to the Hausdorff distance d_H. This quantity seems less stable than the volume of the approximation, where compensation mechanisms might occur around the boundary of K. The problem of assessing the Hausdorff distance between K and K^χ had been raised in [6], and a first result was obtained by Calka and Chenavier [2]. We complete their findings by showing that, under positive inner and outer densities for K, there are constants $\varepsilon_K, \varepsilon'_K > 0$ such that

$$\mathbf{P}\left(\varepsilon_K \leqslant \frac{d_H(K, K_{\chi_\lambda})}{\lambda^{-1/d}\ln(\lambda)^{1/d}} \leqslant \varepsilon'_K\right) \to 1$$

as $\lambda \to \infty$, where χ_λ is a Poisson point process with intensity λ. The upper bound also holds for a binomial point process.

The plan of the paper is as follows. In Section 2, we recall basic facts and definitions about self-similar sets, especially regarding upper and lower Minkowski contents. We then give conditions under which self-similar boundaries have positive inner and outer densities. Voronoi approximation is formally introduced in Section 3. We then derive the volume normal approximation for sets with well-behaved self-similar boundaries and more general Hausdorff distance results. We also develop the counter example K_{cantor} that satisfies neither the hypotheses of Theorem 2 nor the Volume approximation variance asymptotics given above.

2 Self-similar sets

2.1 Self-similar set theory

We start with some brief reminders of self-similar set theory. A precise treatment of the subject can be found in [4]. We recall the definition of the Hausdorff distance between two sets $A, B \subset \mathbb{R}^d$,

$$d_H(A,B) = \inf\{r > 0 : A \subset B + B(0,r),\ B \subset A + B(0,r)\}.$$

Let $\{\phi_i, i \in I\}$ be an iterated function system, i.e a finite set of contracting similitudes. We define the following set transformation

$$\begin{aligned} \psi : \mathcal{P}(\mathbb{R}^d) &\longrightarrow \mathcal{P}(\mathbb{R}^d) \\ E &\longmapsto \bigcup_i \phi_i(E). \end{aligned}$$

ψ is easily seen to be a contracting transformation for the Hausdorff metric, which happens to be complete on $\mathcal{K}^d$, the class of non-empty compact sets of $\mathbb{R}^d$. By a fixed point theorem, there is an unique set $E \in \mathcal{K}^d$ satisfying $\psi(E) = E$, who is by definition the self-similar set associated with the ϕ_i.

If there is a bounded open set U such as $\psi(U) = \bigcup \phi_i(U) \subset U$ with the union disjoint, then necessarily $E \subset \mathrm{cl}(U)$ and the ϕ_i are said to satisfy the *open set condition*. Schief proved in [11] that we can pick U so that $U \cap E$ is not empty. We will always do so here.

The similarity dimension of E is the unique s satisfying

$$\sum \lambda_i^s = 1$$

where λ_i is the stretching factor of ϕ_i. When the open set condition holds, this similarity dimension is also the Hausdorff dimension and the Minkowski dimension of E. Furthermore, E's upper and lower s-dimensional Minkowski contents are finite and positive. This is an easy and probably known result, but since we have not found it explicitly stated and separately proven in the literature, we will do so here in Proposition 1 (one can find an alternative proof for the lower content in [5, Paragraph 2.4], it can also be considered a consequence of $\mathcal{H}^s(E) > 0$, like suggested in [8]). We will need the following classical lemmae, that we prove for completeness.

Lemma 1. *Let (U_i) be a collection of disjoint open sets in $\mathbb{R}^d$ such as each U_i contains a ball of radius $c_1 r$ and is contained in a ball of radius $c_2 r$. Then any ball of radius r intersects at most $(1 + 2c_2)^d c_1^{-d}$ of the sets $\mathrm{cl}(U_i)$.*

Proof. Let B be a ball of center x and radius r. If some $\mathrm{cl}(U_i)$ intersects B then $\mathrm{cl}(U_i)$ is contained in the ball B' of center x and radius $r(1 + 2c_2)$. If q different $\mathrm{cl}(U_i)$ intersect B then there are q disjoint balls of radius $c_1 r$ inside B', and by comparing volumes $q \leqslant (1 + 2c_2)^d c_1^{-d}$. □

Lemma 2. *Suppose E and the ϕ_i satisfy the open set condition with U. Then for every $r < 1$ we can find a finite set $\mathcal{A}$ of similarities Φ_k with ratios Λ_k such as*

1. *The Φ_k are composites of the ϕ_i.*
2. *The $\Phi_k(E)$ cover E.*

3. *The* $\Phi_k(U)$ *are disjoint.*

4. $\sum \Lambda_k^s = 1$ *where* s *is the similarity dimension of* E.

5. $\min_i(\lambda_i)r \leqslant \Lambda_k < r$ *for all* k.

Proof. We give an algorithmic proof. Initialise at step 0 with $\mathcal{A} = \{Id\}$. At step n replace every $\Phi \in \mathcal{A}$ with ratio greater than r by the similarities $\Phi \circ \phi_i, i \in I$. Stop when the process becomes stationary, which will happen no later than step $\lceil \ln(r)/\ln(\max(\lambda_i))\rceil$.

Obviously, point 1 is satisfied. We will prove the next three points by induction. At step 0, all of E is covered by the $\Phi_k(E)$, the $\Phi_k(U)$ are disjoint, and the Λ_k^s sum up to 1. The first property is preserved when Φ is replaced by the $\Phi \circ \phi_i$, since $\Phi(E) = \Phi(\psi(E)) = \bigcup \Phi \circ \phi_i(E)$. Likewise, the $\Phi \circ \phi_i(U)$ are disjoint one from each other because Φ is one-to-one, and disjoint from the other $\Phi_k(U)$ because $\bigcup \Phi \circ \phi_i(U) = \Phi(\psi(U)) \subset \Phi(U)$, which yields point 3. For point 4 note that if Φ has ratio Λ, then the $\Phi \circ \phi_i$ have ratios $\Lambda\lambda_i$ and $\Lambda^s = \Lambda^s \sum \lambda_i^s = \sum (\Lambda\lambda_i)^s$ so the sum of the Λ_k^s remains unchanged by the substitution. Finally, since $r < 1$, every final set of the process has an ancestor with ratio greater than r. This gives the lower bound for point 5; the upper bound comes from the fact that the process ends. □

Remark 1. The process in the proof of Lemma 2 is often resumed as follows

$$\mathcal{A} = \{\phi_{i_1} \circ \phi_{i_2} ... \circ \phi_{i_n} \mid \prod_{k=1}^{n} \lambda_{i_k} < r \leqslant \prod_{k=1}^{n-1} \lambda_{i_k}\}.$$

2.2 Minkowski contents of self-similar sets

We recall that the s-dimensional lower Minkowski content of a non-empty bounded set $E \subset \mathbb{R}^d$ can be defined as

$$\liminf_{r>0} \frac{\mathrm{Vol}(E + B(0,r))}{r^{d-s}}$$

Similarly, the s-dimensional upper Minkowski content of E is

$$\limsup_{r>0} \frac{\mathrm{Vol}(E + B(0,r))}{r^{d-s}}$$

In this paper, when both contents are finite and positive, we will simply say that E has upper and lower Minkowski contents. That leaves no ambiguity on the choice of s, since in that case s is necessarily the Minkowski dimension of E, i.e

$$s = d - \lim_{r \to 0} \frac{\ln(\mathrm{Vol}(E + B(0,r)))}{\ln(r)}.$$

Proposition 1. *Let E be a self-similar set satisfying the open set condition with similarity dimension s. Then E has finite positive s-dimensional upper and lower Minkowski contents, i.e*

$$0 < \liminf_{r>0} r^{d-s} Vol(E + B(0,r)) \leqslant \limsup_{r>0} r^{d-s} Vol(E + B(0,r)) < \infty.$$

Proof. As before, let ϕ_i be the generating similarities of E, λ_i their ratios, $\psi : A \mapsto \bigcup \phi_i(A)$ the associated set transformation, and U the open set of the open set condition. Choose any $0 < r < 1$ and define the Φ_k, Λ_k as in Lemma 2. Finally, write $E_k = \Phi_k(E), U_k = \Phi_k(U)$.

We approximate $E + B(0,r)$ by the sets $E_k + B(0,r)$, who are similar to the $E + \Phi_k^{-1}(B(0,r))$. By construction $\Phi_k^{-1}(B(0,r))$ is a ball with a radius belonging to $[1, (\min_i \lambda_i)^{-1}]$, so that

$$\mathrm{Vol}(B(0,1)) \leqslant \mathrm{Vol}(E + \Phi_k^{-1}(B(0,r))) \leqslant \mathrm{Vol}(B(0, \mathrm{diam}(E) + (\min_i \lambda_i)^{-1})),$$

because E is not empty. Applying Φ_k we get

$$c' \Lambda_k^d \leqslant \mathrm{Vol}(E_k + B(0,r)) \leqslant C \Lambda_k^d$$

for some positive c', C independent from r (the exact value of the constants doesn't matter here).

Since $E + B(0,r) \subset \bigcup_k E_k + B(0,r)$ and $\sum \Lambda_k^s = 1$ we immediately get the upper bound

$$\begin{aligned} \mathrm{Vol}(E + B(0,r)) &\leqslant \sum \mathrm{Vol}(E_k + B(0,r)) \\ &\leqslant \sum C \Lambda_k^d \\ &\leqslant C \sum \Lambda_k^s r^{d-s}. \\ &\leqslant C r^{d-s}. \end{aligned}$$

For the lower bound we apply Lemma 1 to the disjoint U_k. Since U is open we can put some ball of radius c_1 in U, and conversely we can put U in some ball of radius c_2, since U is bounded. This means that each of the U_k contains a ball of radius $r \min_i(\lambda_i) c_1 \leqslant \Lambda_k c_1$ and is contained in a ball of radius $r c_2 \mathrm{diam}(U) \geqslant \Lambda_k c_2$. So for any $x \in E + B(0,r)$, $B(x,r)$ intersects at most q of the E_k (since $E_k \subset \mathrm{cl}(U_k)$) with q a positive integer independent of r and x. This can be rewritten $\mathbf{1}_{E+B(0,r)} \geqslant \dfrac{1}{q} \sum \mathbf{1}_{E_k + B(0,r)}$. Integrating we get $\mathrm{Vol}(E + B(0,r)) \geqslant \dfrac{1}{q} \sum \mathrm{Vol}(E_k + B(0,r))$ so that

$$\begin{aligned}\mathrm{Vol}(E+B(0,r)) &\geqslant \frac{1}{q}\sum \mathrm{Vol}(E_k+B(0,r))\\ &\geqslant \frac{c'}{q}\sum \Lambda_k^d\\ &\geqslant \frac{c'}{q}(\min_i \lambda_i)^{d-s}\sum \Lambda_k^s r^{d-s}\\ &\geqslant cr^{d-s}.\end{aligned}$$

□

2.3 Boundary regularity

In order to formulate our result, we need to introduce the following definition of proper points, not to be mistaken with the *essential points* mentioned in the introduction. Proper points will be further discussed in Section 3.2. We can already note that K must have no unproper points if we want a positive lower bound for the f_r^K on K.

Definition 1. *A point x is said to be a proper point of K if the intersection of K with any neighbourhood of x has positive volume, otherwise it is unproper to K.*

Our main result holds for self-similar subsets E of ∂K satisfying the following assumption:

Assumption 1. *E satisfies the open set condition with some set U(with $U \cap E \neq \emptyset$) such that $U \cap \partial K \subset E$ and $U \setminus \partial K$ has finitely many connected components.*

This assumption can be justified heuristically: if E cuts the space into infinitely many connected components, then because of self-similarity it also does so locally, and K and K^c are too disconnected for a ball to be rolled inside them. Example 2 will show that these concerns are legitimate.

Theorem 2. *Let K be a non-empty compact set with no unproper points and Vol$(\partial K) = 0$. Let E be a self-similar subset of ∂K for which Assumption 1 holds. Then K has a rolling ball along E, i.e there are constants $\delta, \varepsilon > 0$ such that, for all $x \in E, r < \delta$, $B(x,r) \cap K^c$ and $B(x,r) \cap K$ both contain a ball of radius εr.*

Proof. Let ϕ_i be the generating similarities of E, λ_i their ratios, $\psi : A \mapsto \bigcup \phi_i(A)$ the associated set transformation. The V_j are the connected components of $U \setminus E$. Since there are finitely many of them, we can suppose they all contain a ball of radius $\tau > 0$. Suppose $\mathrm{diam}(U) = \mathrm{diam}(\mathrm{cl}(U)) = 1$, pick any $0 < r < 1$ and $x \in E$.

Lemma 2 shows that there is a similarity Φ such that $\min(\lambda_i)r \leqslant \operatorname{diam}(\Phi(U)) < r$ and $x \in \Phi(E)$. It follows that $\Phi(U) \subset B(x,r)$. We also have $\Phi(U) \cap \partial K = \Phi(U) \cap E = \Phi(U \cap E)$. Indeed, for any point x' of E outside $\Phi(E)$ there is another similarity Φ' of Lemma 2 such as $x' \in \Phi'(E)$ and $\Phi'(U) \cap \Phi(U) = \emptyset$, which implies $\mathrm{cl}(\Phi'(U)) \cap \Phi(U) = \Phi'(\mathrm{cl}(U)) \cap \Phi(U) = \Phi'(E) \cap \Phi(U) = \emptyset$ so that $x' \notin \Phi(U)$.

Consequently, for all j, $\Phi(V_j)$ has no intersection with ∂K. So $\Phi(V_j) \cap \mathrm{int}(K)$ and $\Phi(V_j) \cap K^c$ are two disjoint open set sets who cover $\Phi(V_j)$, and we must have either $\Phi(V_j) \subset K$ or $\Phi(V_j) \subset K^c$.

Since there is a point y in $\Phi(U) \cap E$ and K has no unproper points, we must have $\mathrm{Vol}(K \cap U), \mathrm{Vol}(K^c \cap U) > 0$. Because $\mathrm{Vol}(E) = 0$, this can only happen if one of the $\Phi(V_j)$ is included in K^c and another in K. Hence $B(x,r) \cap K$, $B(x,r) \cap K^c$ each contain a ball of radius $\operatorname{diam}(\Phi(U))\tau$. Since $\Phi(U) \geqslant \min(\lambda_i)r$, the rolling ball condition holds with $\varepsilon = \min(\lambda_i)\tau$. □

Remark 2. As we pointed out in the introduction, this implies that K, K^c have lower density bounds on E. More precisely, for appropriate $\delta, \varepsilon > 0$

$$\forall x \in E, r < \delta, \quad f_r^K(x), f_r^{K^c}(x) \geqslant \varepsilon. \tag{1}$$

This weaker statement is enough for our purposes regarding Voronoi approximation.

We show below that the Von Koch flake satisfies the hypotheses of Theorem 2.

Example 1. Let E be the self-similar set associated with the direct similarities $\phi_i : \mathbb{R}^2 \to \mathbb{R}^2$ sending $S = A_0A_4$ to $a_i = A_{i-1}A_i$, for $i = 1, 2, 3, 4$, in the configuration of Figure 1, who satisfy the open set condition with U the interior of the triangle $A_0A_2A_4$. Such sets E are called Von Koch curves. Looking at the iterates $\psi^{(n)}(S)$ in Figure 2 gives an idea of the general form of the Von Koch curve and of why it is said to be self-similar.

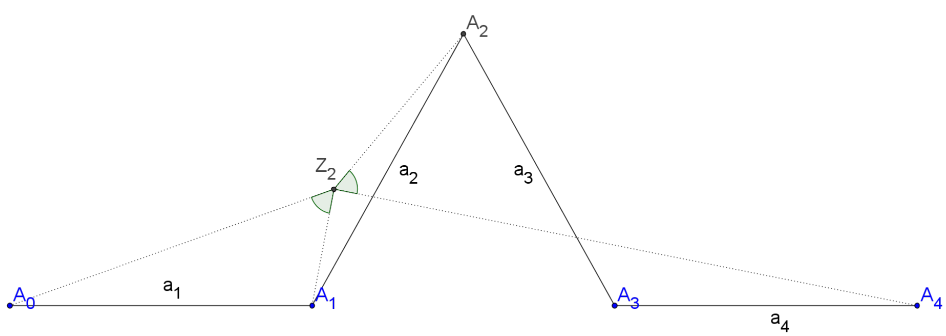

Figure 1: The generating similitudes of the Van Koch curve. Z_2 is the center of the similarity ϕ_2.

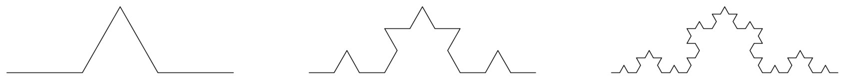

Figure 2: The sets $\psi^{(1)}(S), \psi^{(2)}(S), \psi^{(3)}(S)$.

Note that the $\psi^{(n)}(S)$ are curves, i.e the images of continuous mappings $\gamma_n : [0,1] \to \mathbb{R}^2$. The γ_n can be chosen to be a Cauchy sequence for the uniform distance between curves in $\mathbb{R}^2$. Then their limit γ is also a continuous mapping, $\gamma([0,1])$ is compact and has distance 0 with E in the Hausdorff metric, so $\gamma([0,1]) = E$. This proves that the Van Koch curve is, indeed, a curve. It can also be shown to be a non-intersecting curve (the image of an injective continuous mapping from $[0,1]$ into $\mathbb{R}^2$).

With a similar reasoning, if we stick three Von Koch curves of same size as in Figure 3, we get a closed non-intersecting curve $\mathcal{C}$. Jordan's curve theorem says $\mathbb{R}^2 \setminus \mathcal{C}$ has exactly two connected components who both have $\mathcal{C}$ as topological boundary. The closure K of the bounded component is a compact set with no unproper points satisfying $\partial K = \mathcal{C}$. K is called a Von Koch flake.

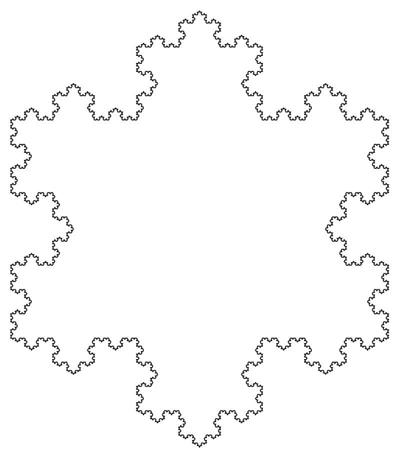

Figure 3: The boundary of the Von Koch flake K.

Now, construct kites C_1, C_2, C_3 on each of the Von Koch curves E_1, E_2, E_3 making ∂K as in Figure 4. It is easy to see that as long as the two equal angles of the lower triangle are flat enough, $C_i \cap \partial K = C_i \cap E_i$. Furthermore, applying Jordan's curve theorem to the E_i and the two upper (resp. lower) edges of the corresponding C_i shows that the $C_i \setminus E_i$ have exactly two connected components.

Consequently, Theorem 2 can be applied three times to obtain a lower bound for f_r on ∂K.

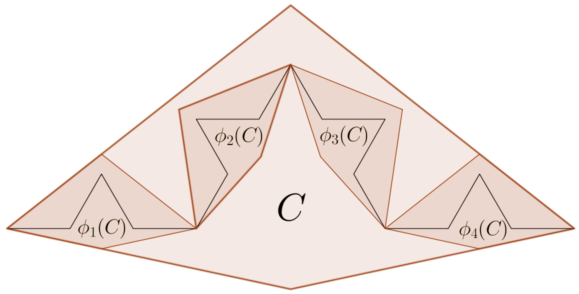

Figure 4: Assumption 1 is satisfied with the kite C.

3 Voronoi approximation

In this paper, χ is a locally finite point process. If $\chi = \chi_n = \{X_1, X_2, ..X_n\}$, where the X_i are iid random points uniformly distributed over $[0,1]^d$, we speak of binomial input; if $\chi = \chi'_n$ is a homogenous Poisson point process of intensity n we speak of Poisson input. We also suppose from now on that K is a subset of $[0,1]^d$.

On the unit cube, define the Voronoi cell $\upsilon_\chi(x)$ of nucleus x with respect to χ as the set of points closer to x than to χ

$$\upsilon_\chi(x) = \{y \in [0,1]^d : \forall x' \in \chi, d(x,y) \leqslant d(x',y)\}$$

The Voronoi approximation K_χ of K is the closed set of all points which are closer to $K \cap \chi$ than to $K^c \cap \chi$. Its name comes from the relation

$$K_\chi = \bigcup_{x \in \chi \cap K} \upsilon_\chi(x).$$

Vol(K_{χ_n}) can be given as an unbiased estimator for Vol(K), it converges almost surely for binomial input as proved by Penrose in [9] as $n \to \infty$. Some papers with more precise results about the order of magnitude of the variance and central limit theorems dealt with the case where K is convex, see for example [6] and the references given in the Introduction.

3.1 Asymptotic normality

We recall below the results of [7], conditions on K that ensure that with binomial input, the volume approximation is asymptotically normal when the number of points tends to ∞. We furthermore give the variance magnitude and an upper bound on the speed of convergence for the Kolmogorov distance. The regularity of the boundary is essential to have a matching lower bound on the variance and a good rate of convergence, but it is still possible to have a bound in the case where the set does not satisfy the conditions below. For all $r > 0$ define

$$\begin{aligned} \partial K_r &= \partial K + B(0,r) = \{x \in \mathbb{R}^d : d(x,K) \leqslant r\}, \\ \partial K_r^+ &= \partial K_r \cap K^c, \\ \partial K_r^- &= \partial K_r \cap K. \end{aligned}$$

We now explicitly state the boundary regularity assumption made on K. It is a weakened form of the rolling ball condition.

Definition 2 (Weak rolling ball condition)**.** *A set K with no unproper points satisfies the weak rolling ball condition whenever*

$$\liminf_{r>0} \frac{1}{Vol(\partial K_r)} \left(\int_{\partial K_r^+} (f_r^K(x))^2 dx) + \int_{\partial K_r^-} (f_r^{K^c}(x))^2 dx) dx \right) > 0. \tag{2}$$

If ∂K has upper and lower Minkowski contents, this last assertion is equivalent to the apparently weaker one

$$\liminf_{r>0} \frac{1}{Vol(\partial K_r)} \left(\int_{\partial K_r^+} (f_{Cr}^K(x))^2 dx) dx) + \int_{\partial K_r^-} (f_{Cr}^{K^c}(x))^2 dx) dx \right) > 0 \text{ for some positive } C. \tag{3}$$

Proof. The first condition obviously implies the second. Now suppose the second condition is satisfied for some $C > 0$. If $C < 1$ then the inequality $f_r^K \geqslant f_{Cr}^K C^{-d}$ and its counterpart for $f_r^{K^c}$ show that (2) holds. If $C > 1$ we can replace ∂K_r by ∂K_{Cr} in the domains of the integral, then divide by $\mathrm{Vol}(\partial K_{Cr})$ instead of $\mathrm{Vol}(\partial K_r)$ and put $r' = Cr$. We're back to (2) and the $\liminf$ is still be positive, since the first operation only made the integrals bigger, and ∂K's Minkowski contents put a lower bound on $\dfrac{\mathrm{Vol}(\partial K_r)}{\mathrm{Vol}(\partial K_{Cr})}$. □

If the lower density bounds of (1) hold, then (2) holds as well with the left hand being greater than ε^2. We can now reproduce below the result derived in [7] for Voronoi approximation.

Theorem 3. *(Lachieze-Rey - Peccati) Let K be a subset of $[0,1]^d$. Assume that for some $s \geq 0$*

$$0 < \liminf_{r>0} r^{s-d} Vol(\partial K^r) \leqslant \limsup_{r>0} r^{s-d} Vol(\partial K^r) < \infty, \tag{4}$$

and that K satisfies the weak rolling ball condition (Definition 2), then

$$0 < \liminf_{r>0} \frac{\mathbf{Var}(Vol(K_{\chi_n}))}{n^{-2+s/d}} \leqslant \limsup_{r>0} \frac{\mathbf{Var}(Vol(K_{\chi_n}))}{n^{-2+s/d}} < \infty, \tag{5}$$

and for all $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that for all $n \geqslant 1$

$$\sup_{t \in \mathbb{R}} \left| \mathbf{P} \left(\frac{Vol(K_{\chi_n}) - \mathbf{E} Vol(K_{\chi_n})}{\mathbf{Var}\left(Vol(K_{\chi_n})\right)} \leqslant t \right) - \mathbf{P}(N \leqslant t) \right| \leqslant C_\varepsilon n^{-s/2d+\varepsilon}. \tag{6}$$

The consequences of Theorems 2 and 3 for sets K with self-similar boundary are immediate. In such a case, the Minkowski dimension s of ∂K is linked to the α above by $s = d - \alpha$, and condition (4) automatically holds by Proposition 1.

Corollary 1. *Let K be a compact set such that ∂K is a finite union of self-similar sets satisfying Assumption 1. Then (5) and (6) hold.*

This corollary applies to the Von Koch flake with $s = \ln(4)/\ln(3)$ (Example 1). The conclusions of Theorem 3 also apply for instance to the Von Koch anti flake, where three Von Koch curves are sticked together like for building the flake, but here the curves are pointing inwards, and not outwards (Figure 5). Assumption 1 is not satisfied on the whole boundary, but it is within an open ball of $\mathbb{R}^d$ intersecting one and only one of the three curves, and having (1) on a self-similar E with same Minkowski dimension as ∂K is actually enough for the weak rolling ball condition to hold.

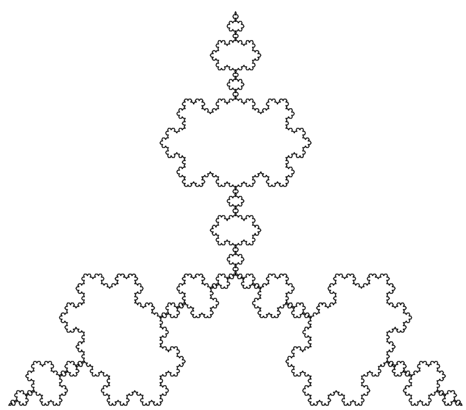

Figure 5: The Von Koch antiflake

In Section 3.3 we exhibit an example of a set K such that ∂K is self-similar and K does not satisfy Assumption 1. We also run simulations suggesting that (5) is also false. Our theorem gives a set of sufficient conditions, but other versions should be valid. For instance, the question of whether a compact set $K \subset \mathbb{R}^2$ whose boundary is a locally self-similar Jordan curve satisfies the conclusions of the theorem above seems to be of interest.

3.2 Convergence for the Hausdorff distance

In this section we will make use of r-coverings and r-packings. Let $\mathcal{B}$ be a collection of balls having radius r and centers belonging to some set E. $\mathcal{B}$ is said to be an r-packing of E if the balls are disjoint. It is an r-covering if the balls cover E.

Coverings and packings are closely related to the Minkowski dimension of E. In particular, if E has upper and lower Minkowski contents, then for r small enough we can find r-coverings of E and $E + B(0, r)$ with less than Cr^{-s} balls, and r-packings of the same sets with more than cr^{-s} balls, where c and C are positive constants not depending on r. More precise results can be found in [8]. The following geometrical lemma shows a connection between coverings, packings, and the Hausdorff distance with the Voronoi approximation.

Lemma 3. *Let K be a non-empty compact set and ∂K its topological boundary.*

1. *Consider a $r/2$-covering $\mathcal{B}$ of both ∂K_r and ∂K_r^-. If the interior of every ball from $\mathcal{B}$ contains a point from χ, and every ball from $\mathcal{B}$ centered on K contains a point from $\chi \cap K$, then $d_H(K, K_\chi) \leqslant r$.*

2. *Consider a $3r$-packing $\mathcal{B}$ of ∂K. If for some ball of $\mathcal{B}$ centered on $x \in \partial K$ we have $\chi \cap B(x, 3r) \cap K = \emptyset$ and $\chi \cap B(x, r) \cap K^c \neq \emptyset$, then $d_H(K, K_\chi) \geqslant r$.*

Proof. We begin with the first point. Let us prove that for all $x \in K \setminus \partial K_r$ we have $x \in K_\chi$. Indeed, if this were not the case, there would be a point $c_x \in \chi \cap K^c$ such as $x \in \upsilon_\chi(c_x)$. The segment joining c_x and x must contain a point from ∂K. Let x_0 be the point of ∂K_r closest to x on that segment. We must have $d(x_0, \partial K) = r$ and $x_0 \in K$ since otherwise there would be another point of ∂K_r closer to x. As a consequence $d(x_0, c_x) > r$. But then by hypothesis there is a ball of $\mathcal{B}$ who contains x_0 along with a point of χ. So c_x isn't the point of χ closest to x, and we have a contradiction. Similarly we can show that $K^c \setminus \partial K_r \subset K_\chi^c$, which reformulates as $K_\chi \subset K + B(0, r)$.

To have $d_H(K, K_\chi) \leqslant r$ it is enough to show that $K \subset K_\chi + B(0, r)$. Let x be a point of K. We just showed that if $x \notin \partial K_r$ then $x \in K_\chi$. And if $x \in \partial K_r$ then by hypothesis there is a ball of $\mathcal{B}$ centered on K with a point of $K \cap \chi$ inside. In all cases $x \in K_\chi + B(0, r)$.

Now we prove the second point. Let y be a point of $\chi \cap B(x, r) \cap K^c$. Then all of the points in $B(x, r)$ are closer to y than to the points outside of $B(x, 3r)$. Consequently all points $B(x, r)$ must lie in Voronoi cells centered in K^c, and $x \notin K_\chi + r'$ for all $r' \leqslant r$, so that $d_H(K, K_\chi) \geqslant r$. □

To formulate results regarding the Hausdorff distance between K and K_χ, the concept of proper points (as introduced in Definition 1) proves to be useful. Unproper points are 'forgotten' by the Voronoi approximation K_χ of K. Though that has no incidence when measuring volumes, it becomes a nuisance when measuring Hausdorff distances.

Let us call proper part of K the set K^{prop} of points proper to K. K^{prop} can be thought of as the complement of the biggest open set O such as $\mathrm{Vol}(O \cap K) = 0$, from which it follows that K^{prop} is compact and that $K_\chi = K_\chi^{prop}$ a.s.

Proposition 2. $K_{\chi_n} \underset{n \to +\infty}{\longrightarrow} K^{prop}$ *almost surely in the sense of the Hausdorff metric for both Poisson and binomial input.*

Proof. Since $K_\chi = K_\chi^{prop}$ almost surely this is equivalent to the fact that $K_{\chi_n} \to K$ almost surely when K has no unproper points. By the Borel-Cantelli lemma it is enough to show the series $\sum_{n \geqslant 1} \mathbf{P}(d_H(K_{\chi_n}, K) > r)$ is convergent for any positive r.

Cover ∂K_r as in point 1 of the previous lemma. Since K is bounded, this can be done with finitely many balls. Let V be the minimum of $\mathrm{Vol}(K \cap B)$ over the $B \in \mathcal{B}$ centered on K. Because K has no unproper points, $V > 0$. The probability of having $d_H(K_{\chi_n}, K) > r$ is at most that of the requirements of point 1 not being satisfied. The latter is bounded by $|\mathcal{B}|(1-V)^n$ for binomial input and $|\mathcal{B}|e^{-nV}$ for Poisson input. In all cases the series associated with $\mathbf{P}(d_H(K_{\chi_n}, K) > r)$ converges, as required. $\square$

The following result gives an order of magnitude for $d_H(K, K_\chi)$ with Poisson input, under assumptions on ∂K, f_r^K and $f_r^{K^c}$ resembling those of Theorem 3.

For ease of notation, we put $f_r = f_r^K$ and $g_r = f_r^{K^c}$.

Theorem 4. *Suppose that ∂K has Minkowski dimension $s > 0$ with upper and lower contents, and that for all r small enough and $x \in K$,*

$$f_r(x) \geqslant \varepsilon_f,$$
$$g_r(x) \geqslant \varepsilon_g,$$

then

$$\mathbf{P}\left(\alpha \leqslant \frac{d_H(K, K_{\chi_\lambda})}{(\lambda^{-1}\ln(\lambda))^{1/d}} \leqslant \beta\right) \underset{\lambda \to +\infty}{\longrightarrow} 1$$

where the χ_λ are Poisson point process of intensity λ and

$$\alpha < (1/3)(s/d)^{1/d}(\kappa_d - \varepsilon_g)^{-1/d}$$
$$\beta = 2(s/d)^{1/d}\varepsilon_f^{-1/d}.$$

Proof. We start with the upper bound. For all λ let A_λ be the event where all the requirements from point 1 of Lemma 3 are met for $\chi = \chi_\lambda$, $r = r_\lambda = \beta(\lambda^{-1}\ln(\lambda))^{1/d}$, $\mathcal{B} = \mathcal{B}_\lambda$ with $\mathcal{B}_\lambda$ having $\mathcal{N}_\lambda = O(r_\lambda^{-s})$ balls of radius $r_\lambda/2$. Since $A_\lambda \subset \{d_H(K, K_\chi) \leqslant r_\lambda\}$ we only need to show that $\lim\limits_{\lambda \to +\infty} \mathbf{P}(A_\lambda) = 1$.

For λ big enough, the probability for a ball $B \in \mathcal{B}_\lambda$ centered on K not having a point from $\chi_\lambda \cap K$ is at most

$$\exp(-\lambda\varepsilon_f 2^{-d} r_\lambda^d) = \lambda^{-s/d}.$$

This bound also works for the probability of a ball $B \in \mathcal{B}_\lambda$ not having a point from χ_λ, so we have

$$\mathbf{P}(A_\lambda^c) \leqslant N_\lambda \lambda^{-s/d}.$$

The right hand has order $\ln(\lambda)^{-s/d}$, so it goes to 0 with λ.

The proof for the lower bound is quite similar. Fix some $\delta > 0$, and redefine A_λ to be the event where the requirements described in point 2 of Lemma 3 are met for $\chi = \chi_\lambda$, $r = r_\lambda = \alpha(\ln(\lambda)\lambda^{-1})^{1/d}$ and $\mathcal{B} = \mathcal{B}_\lambda$ a collection of $\mathcal{N}_\lambda \geqslant c r_\lambda^{-s}$ balls having radius $3r_\lambda$. We have $A_\lambda \subset \{d_H(K, K_\chi) \geqslant r_\lambda\}$.

The probability of there being no points of $K \cap \chi_\lambda$ in a ball $B(x, 3r_\lambda) \in \mathcal{B}_\lambda$ and at least one point of $K^c \cap \chi_\lambda$ in $B(x, r_\lambda)$ is

$$\exp(-\lambda(\kappa_d - g_{3r_\lambda}(x))3^d r_\lambda^d)(1 - \exp(-\lambda r_\lambda^d g_{r_\lambda}(x))).$$

Consequently, we have the following upper bound for λ big enough

$$\mathbf{P}(A_\lambda^c) \leqslant (1 - e^{-\lambda(\kappa_d - \varepsilon_g)3^d r_\lambda^d}(1 - e^{-\lambda r_\lambda^d \varepsilon_g}))^{\mathcal{N}_\lambda}.$$

We would like the right hand to go to 0 with λ. Taking its logarithm this is equivalent to

$$\mathcal{N}_\lambda \exp(-\lambda(\kappa_d - \varepsilon_g)3^d r_\lambda^d)(1 - \exp(-\lambda r_\lambda^d \varepsilon_g)) \underset{\lambda \to +\infty}{\longrightarrow} +\infty$$

Because $\exp(-\lambda(\kappa_d - \varepsilon_g)3^d r_\lambda^d) = \lambda^{\delta - s/d}$ with $\delta > 0$, it is indeed the case.

□

The proof and the result call for some comments.

Remark 1. The Minkowski dimension of ∂K has no impact on the order of magnitude of $d_H(K, K_\chi)$. In fact, if ∂K doesn't have a Minkowski dimension, we can do the coverings with $O(r^{-d})$ balls, so the upper bound still holds after replacing s by d in the expression of β. This is the result given by Calka and Chenavier in [2], with a slightly better constant. If ∂K's so-called lower Minkowski dimension is positive, given any $\epsilon > 0$, we can do the packing with $r^{-s+\epsilon}$ balls for λ big enough, and also get a lower bound. This is automatically the case when $d \geqslant 2$ and K has positive volume.

Remark 2. If $s = 0$ and ∂K has Minkowski contents then actually ∂K has a finite number of points and $d_H(K, K_{\chi_\lambda})$ has order $\lambda^{-1/d}$ in the sense that

$$\mathbf{P}(d_H(K, K_{\chi_\lambda})\lambda^{1/d} > t) \leqslant \exp(-\varepsilon_f (t/2)^d),$$

which is enough to guarantee the existence of moments of all orders for $d_H(K, K_{\chi_\lambda})\lambda^{1/d}$. If we don't have Minkowski contents the situation might be more delicate.

Remark 3. For binomial input, some minor changes in the proof give the same upper bound. It can't be done for the lower bound since we use the fact that $\chi \cap A, \chi \cap B$ are independent when A and B are disjoint and χ is a Poisson point process.

Remark 4. The upper bound we give is sensible to the value of ε_f, and we need to have $f_r \geqslant \varepsilon_f$ on all of K. On the contrary, requesting only that $g_r \geqslant \varepsilon_g$ on a subset of ∂K having positive Minkowski dimension we could still give a lower bound, and we can choose that bound to be independent from ε_g. Heuristically, this is because bounding $d_H(K, K_\chi)$ from above requires χ to be well enough distributed along all of ∂K, whereas an isolated incident is enough to provide a lower bound.

Theorem 4 requires a lower bound for f_r on all of K. The following lemma shows that it is enough to find one on ∂K. As a consequence, Theorem 4 holds under the conditions of Corollary 1.

Lemma 4. *If for all $r < \delta$ we have $f_r \geqslant \varepsilon$ on ∂K, then for all $r < \delta$ we have $f_r \geqslant \frac{\varepsilon}{2^d}$ on K.*

Proof. If x is in $\partial K_{r/2}$ then $B(x,r)$ contains a ball of radius $r/2$ centered on $x' \in \partial K$, so $f_r(x) = \mathrm{Vol}(K \cap B(x,r))\kappa_d^{-1} r^{-d} \geqslant \mathrm{Vol}(K \cap B(x', r/2))\kappa_d^{-1} r^{-d} \geqslant \varepsilon 2^{-d}$. If not then the ball $B(x, r/2)$ is contained in K so that $f_r \geqslant 2^{-d} \geqslant \varepsilon 2^{-d}$. □

3.3 A counter-example

Here we construct a set K_{cantor} with self-similar boundary not satisfying the weak rolling ball condition. This example shows that Theorem 2 cannot be generalised by dropping Assumption 1, even if the conclusion is weakened.

The example K below is uni-dimensional, but a counter-example in dimension 2 can be obtained by considering $K \times [0,1]$.

Example 2. Let $E \subset \mathbb{R}$ the self-similar set generated by the similarities $\phi_1 : x \mapsto x/3$, $\phi_2 : x \mapsto (2+x)/3$ who satisfy the open set condition with $U = (0,1)$. E is in fact the Cantor set, and can be characterized as the set of points having a ternary expansion with no ones.

To every positive integer n associate the sequence s'^n of its digits in base 2 in reverse order and double the terms to get s^n. For example, since 6 is 110 in base 2, $s^6 = (0,2,2)$. This defines a bijection between $\mathbb{N}$ and the set of finite sequences of zeroes and twos ending in 2, with the additional property that s^n always has length $l_n \leqslant n$. Now for all n define

$$\begin{aligned} a_n &= \frac{1}{3^{n+1}} + \sum_{k \geqslant 1} \frac{s_k^n}{3^k} \\ b_n &= \frac{2}{3^{n+1}} + \sum_{k \geqslant 1} \frac{s_k^n}{3^k} \\ A_n &= (a_n, b_n) \end{aligned}$$

We have the following ternary expansions

$$\begin{aligned} a_n &= 0.s_1^n s_2^n ... s_{l_n}^n 000...01 \\ &= 0.s_1^n s_2^n ... s_{l_n}^n 000...0022222... \\ b_n &= 0.s_1^n s_2^n ... s_{l_n}^n 000...02 \end{aligned}$$

Now, set $K = \mathrm{cl}(\bigcup A_n)$. We claim that K has no unproper points, $\partial K = E$ and that K does not satisfy the regularity condition of Theorem 3.

Proof. The first assertion is easy to prove. Being segments, the A_n have no unproper points to themselves, so $\bigcup A_n \subset K^{prop}$ and $K \subset K^{prop}$ by taking closures.

For the second assertion we need to show that $\partial K = K \setminus \bigcup A_n = \mathrm{cl}(\bigcup\{a_n, b_n\})$. We already have the obvious $\partial K \subset K \setminus \bigcup A_n$. Define

$$\begin{aligned}
a'_n &= \frac{1}{3^{n+1}} - \frac{2}{3^{l_n}} + \sum_{k\geqslant 1} \frac{s^n_k}{3^k} \\
b'_n &= \frac{2}{3^{n+1}} - \frac{2}{3^{l_n}} + \sum_{k\geqslant 1} \frac{s^n_k}{3^k} \\
A'_n &= (a'_n, b'_n)
\end{aligned}$$

Since for all $n, s^n_{l_n} = 2$, the corresponding ternary expansions are

$$\begin{aligned}
a'_n &= 0.s^n_1 s^n_2 ... s^n_{l_n-1}000...01 \\
&= 0.s^n_1 s^n_2 ... s^n_{l_n-1}000...0022222... \\
b'_n &= 0.s^n_1 s^n_2 ... s^n_{l_n-1}000...02
\end{aligned}$$

If $x \in A_i \cap A'_j$ then every ternary expansion of x has the same digits as the finite ternary expansions of a_i, a'_j up to the first 1, which is impossible. So $\bigcup A'_n$ is an open set disjoint from $\bigcup A_n$ and hence from K. Furthermore, $\bigcup A'_n$ is dense near the a_n, because for all $k, N \in \mathbb{N}^*$, we can find an $a'_{k'}$ whose ternary expansion has the same N first digits as the non-terminating expansion of a_k, so that $d(a_k, a'_{k'}) \leqslant 1/3^N$. A similar argument works for the b_n, so that the a_n, b_n belong to ∂K and, since the latter is closed, $\mathrm{cl}(\bigcup\{a_n, b_n\}) \subset \partial K$.

Finally, consider a point $x \in K \setminus \bigcup A_n$. For all $r > 0$, $B(x, r)$ contains a point from an A_k, and since $x \notin A_k$, one of the two points a_k, b_k must also be in $B(x, r)$. Consequently, x is also an accumulation point of $\bigcup\{a_n, b_n\}$. We just proved that $K \setminus \bigcup A_n \subset \mathrm{cl}(\bigcup\{a_n, b_n\})$. Putting this together with the previous two inclusions we get the desired equality.

Since for all $x \in E, N \in \mathbb{N}^*$ we can find an a_k with the same first N digits as x in base 3, the a_n are dense in E and $E \subset \partial K$. Conversely, $\partial K \subset E$, since the a_n, b_n belong to E, who is closed.

For the last assertion, pick any $r > 0$ and set $N = 2\lceil -\log_3(r)\rceil$. Let X be the union of the balls of radius r centered on the endpoints of the N first A_n. X has area at most $-4r\log_3(r)$ and for any $x \in \partial K_r \setminus X$, $B(x, r)$ does not intersect the $A_k, k \leqslant N$. Since $\mathrm{Vol}(\partial K) = 0$

$$\mathrm{Vol}(K \setminus (A_1 \cup A_2..A_N)) = \mathrm{Vol}(\bigcup_{n>N} A_n) = \frac{1}{2.3^{N+1}} \leqslant r^2.$$

As a consequence $f_r \leqslant r$ on $\partial K_r \setminus X$, so that

$$\int_X \max(f_r^2, g_r^2) \leqslant -4r\log_3(r)$$
$$\int_{\partial K_r^+ \setminus X} f_r^2(x)dx \leqslant \mathrm{Vol}(\partial K_r) r^2$$
$$\int_{\partial K_r^- \setminus X} g_r^2(x)dx \leqslant r^2.$$

After dividing by $\mathrm{Vol}(\partial K_r)$, who has order $r^{1-\ln(2)/\ln(3)}$, the sum of the left terms is bigger than the expression inside the limit of (2), and the sum of the right terms goes to 0 with r. $\square$

Simulations were made for the quality of the Voronoi volume approximation with this set K. The magnitude order of the empirical variance of $\mathrm{Vol}(K_{\chi_n})$ seems to be n^τ with $\tau \approx -1.8$, as shown in Figure 6. Looking at Theorem 3, the approximation behaves as if the set had a "nice" fractal boundary of dimension ≈ 0.2, whereas its real fractal dimension is $1 - \ln(2)/\ln(3) \approx 0.37$.

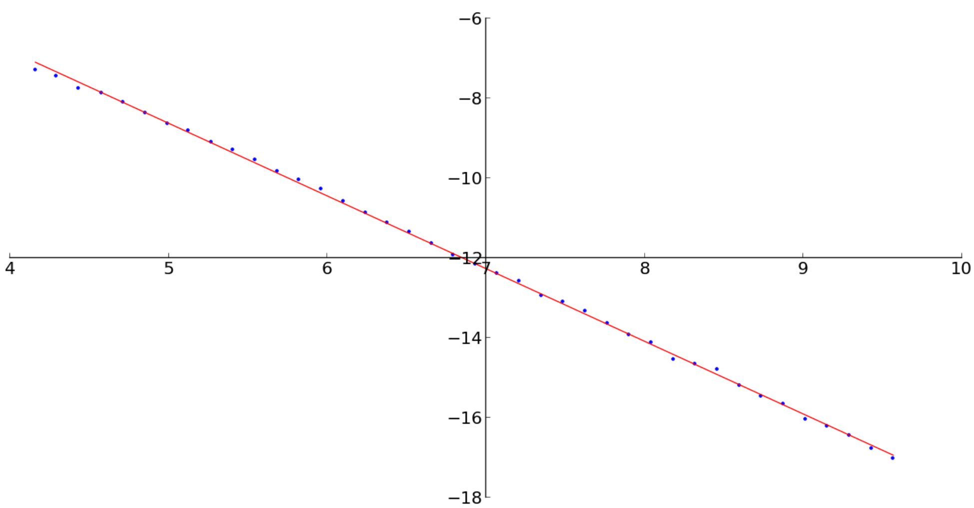

Figure 6: In blue $\ln(\mathbf{Var}(K_{\chi_n}))$ as a function of $\ln(n)$, in red the associated linear regression. For each n, the variance was estimated with 1000 realisations of $\mathrm{Vol}(K_{\chi_n})$.

Simulations also suggest that a central limit theorem still holds. Such a fact indicates that though the results of Lachieze-Rey and Peccati [7] seem to be generalisable, the variance of $\mathrm{Vol}(K_{\chi_n})$ is indeed related to the behaviour of f_r and g_r near ∂K.

Example 3. It is possible to construct other sets not satisfying the regularity condition 2. If we don't require ∂K to be a self-similar set, a much simpler example is given by

$$K = \mathrm{cl}(\bigcup_{n \in \mathbb{N}^*} \left(\frac{1}{n} - \frac{1}{3^n}, \frac{1}{n}\right)).$$

Intentionally, ∂K looks like the set $\{n^{-1}, n \in \mathbb{N}^*\}$, who is often given as an example of a countable set with positive Minkowski dimension. K has Minkowski dimension 1/2 with upper and lower contents, no unproper points, and does not satisfy the requirements of 2. This can be proved using the same methods as in Example 2. Again, simulations tend to show that the variance of $\mathrm{Vol}(K_{\chi_n})$ is about n^τ with $\tau \approx -1,8$ and that a central limit theorem still holds.

References

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Science Publications, 2000.

[2] P. Calka and N. Chenavier. Extreme values for characteristic radii of a Poisson-Voronoi tessellation. arXiv:1304.0170, 2013.

[3] J. H. J Einmahl and E. V. Khmaladze. The two-sample problem in and measure-valued martingales. *Lecture Notes-Monograph Series*, pages 434–463, 2001.

[4] K. J. Falconer. *The geometry of fractal sets.* Cambridge University Press, 1985.

[5] D. Gatzouras. Lacunarity of self-similar and stochastically self-similar sets. *Trans. AMS*, 352(5):1953–1983, 2000.

[6] M. Heveling and M. Reitzner. Poisson-Voronoi approximation. *The Annals of Applied Probability*, pages 719–736, 2009.

[7] R. Lachièze-Rey and G. Peccati. New Kolmogorov bounds for geometric functionals of binomial point processes. in preparation.

[8] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge University Press, 1995.

[9] M. D. Penrose. Laws of large numbers in stochastic geometry with statistical applications. *Bernoulli*, 13(4):1124–1150, 2007.

[10] M. Reitzner, Y. Spodarev, and D. Zaporozhets. Set reconstruction by voronoi cells. *Advances in Applied Probability*, 44(4):938–953, 2012.

[11] A. Schief. Separation properties for self-similar sets. *Proc. Amer. Math. Soc.*, 122(1):111–115, 1994.