Random ergodic theorems and regularizing random weights.

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Abstract

We study the convergence of pointwise ergodic means for random subsequences, in a universal framework, together with ergodic means which are modulated by random weights. The methods used in this work mainly involve Gaussian tools, transference principles and new results on oscillation functions.

1 Introduction

In this work we introduce different and complementary approaches in the study of convergence of pointwise ergodic means for random subsequences, in a universal framework. We also study the pointwise convergence of ergodic means which are modulated by random weights, for any subsequence.

The different strategies consist, for some cases, in using a transference method on $\mathbb{Z}^d$ with the shift, and in showing maximal inequalities (see Section 3). For other cases, we work directly, for instance, on the dynamical system, to show maximal inequalities, or to calculate explicit bounds of the oscillation function associated with the studied transformation (see Section 4). Finally, in Section 5, we study weighted ergodic theorems and we show the regularizing effect of some random weights. In all these cases, the basic tools (classic and less known) will be the spectral lemma and the dilation theorem. We also use results of R. Jones about oscillation in ergodic theory, together with tools which come from the Fourier analysis and the study of regularity of Gaussian random functions'trajectories and which are already known in this domain (see [16]). Section 2 is devoted to these tools.

The first purpose of this work is to give conditions on the process $\{X_k, k \in \mathbb{N}\}$ defined on a complete probability space $(\Omega, \mathcal{B}, P)$ that we assume to be integer-valued, to be able to construct a universal measurable set Ω_0 such that $P(\Omega_0) = 1$ in order that there exist a positive constant C such that for all $\omega \in \Omega_0$, $C(\omega) < \infty$, for any measurable dynamical system $(Y, \mathcal{A}, \mu, T)$ (that is to say, for any probability space $(Y, \mathcal{A}, \mu)$ and any μ-preserving bijective transform $T : Y \to Y$), for all $f \in L^2(\mu)$, we have, on Ω_0,

$$\left\| \frac{1}{N} \sum_{k=1}^{N} f \circ T^{X_k} - \frac{1}{N} \sum_{k=1}^{N} \mathbb{E} f \circ T^{X_k} \right\|_{L^2(\mu)} \le C \parallel f \parallel_{L^2(\mu)} a_N, \qquad (1)$$

with $\lim_{N \to \infty} a_N = 0$.

Hence, by studying the convergence properties of the sequence of deterministic ergodic averages below

$$\left\{ \frac{1}{N} \sum_{k=1}^{N} \mathbb{E} f \circ T^{X_k},\ N \ge 1 \right\}$$

we obtain quantitative and universal results for the convergence of the following random ergodac averages:

$$\left\{ \frac{1}{N} \sum_{k=1}^{N} f \circ T^{X_k(\omega)},\ N \ge 1 \right\}$$

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In the sequel, $\mathbb{E}$ will always denote the symbol of integration on the space $(\Omega, \mathcal{B}, P)$. De plus dans la suite pour tout $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$ et $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{R}^d$, nous notons $< \alpha, \beta >= \sum_{j=1}^d \alpha_j.\beta_j$.

In order to study (1) we shall use essentially spectral tools together with regularity properties of trajectories of Gaussian random functions. More precisely, we shall show the following result qui se presente comme une extention des inegalites de Salem-Zygmung (1954, voir []). En effet, ces dernieres donnent des estimations asymptotique de l'ordre de grandeur uniforme de polynomes trigonometriques aleatoires pour pour un processus independant $\{X_k,\ k \in \mathbb{N}\}$ prennant un nombre fini de valeurs entieres (voir [14]). Dans ce cas leurs preuves reposent sur l'inegalite de Berstein pour les polynomes. Nous proposons de donner une generalisation de ces inegalites; les outils gaussiens remplaceront l'inegalite de Berstein. Nous pourrons ainsi obtenir des estimations asymptotiques de polynomes trigonometriques pour des processus alatoires $\{X_k,\ k \in \mathbb{N}\}$a valeurs dans $\mathbb{R}^d$ possedant une condition de moment. Bien sur pour les applications en theorie ergodique, le processus aleatoire sera a valeurs dans $\mathbb{N}^d$

Theorem 1 *Let $\{X_k,\ k \in \mathbb{N}\}$ be a m-dependent ($m \in \mathbb{N}$, see Definition 2) random process with values in $\mathbb{R}^d$ $(d \geq 1)$ and defined on a complete probability space $(\Omega, \mathcal{B}, P)$. Let $\mathcal{N} \subset \mathbb{N}$ and assume that there exists an increasing positive sequence $(q_N)_{N\in\mathbb{N}}$ such that $(\sqrt{N}/q_N)_{N\geq 1}$ is non increasing sequence and*

$$\frac{1}{q_N} = O\left(\frac{1}{\sqrt{N}\ \log N}\right), \tag{2}$$

and

$$\forall 1 \leq i \leq d, \qquad \mathbb{E} \sup_{N\geq 1} \sqrt{\frac{N\ \log^+(\mid X_N^{(i)} \mid)}{q_N^2}} < \infty, \tag{3}$$

where the symbol $X_N^{(i)}$ denotes the i^{st} component of the random vector X_N. Then we have the following asymptotic estimation

$$\mathbb{E} \sup_{N\in\mathcal{N}} \sup_{\alpha\in[0,1]^d} \left| \frac{1}{q_N} \sum_{k=1}^N \exp\ 2i\pi < \alpha, X_k > - \mathbb{E} \exp\ 2i\pi < \alpha, X_k > \right| < \infty. \tag{4}$$

Notice that the condition (3) is not empty since it contains the following example: $X_k = u_k + \theta_k$ where (u_k) is any strictly increasing sequence of positive integers $(d = 1)$ and (θ_k) a sequence of independent and identically distributed, random variables satisfying

$$\exists \delta > 0/\ \mathbb{E} \mid \theta_1 \mid^\delta < \infty.$$

We can show that the condition is satisfied when $q_N = \sqrt{N\ \log(u_N)}$ [15].

Moreover if we denote by $||.||$ a norm on $\mathbb{R}^d$ and suppose that for some $\delta \in]0,1]$ there exists $\gamma \in]0,1[$ such that $\mathbb{E}\left(||X_N||^\delta\right) = O(2^{N^\gamma})$ then we will prove in 3.1 that the assumptions of theorem 1 are holds with $q_N = N^{\gamma'}$ o $\gamma' \in]\frac{\gamma+1}{2}, 1[$.

Now, from the spectral lemma, we have

$$\left|\left| \frac{1}{N} \sum_{k=1}^N f \circ T^{X_k(\omega)} - \frac{1}{N} \sum_{k=1}^N \mathbb{E} f \circ T^{X_k} \right|\right|_{L^2(\mu)}^2 =$$

$$\int_{[0,1[^d} \left| \frac{1}{N} \sum_{k=1}^N \exp\ 2i\pi < \alpha, X_k(\omega) > - \frac{1}{N} \sum_{k=1}^N \mathbb{E}\ \exp\ 2i\pi < \alpha, X_k > \right|^2 \mu_f(d\alpha), \tag{5}$$

where μ_f is the spectral measure of the operator T at the point f. Theorem 1 gives us therefore a bound for (5) as follows

$$(5) \leq \| f \|_{L^2(\mu)}^2 \left(\frac{q_N}{N}\right)^2 \chi^2(\omega).$$

with

$$\chi(\omega) = \sup_{N\geq 1} \sup_{\alpha\in[0,1]^d} \left| \frac{1}{q_N} \sum_{k=1}^N \exp\ 2i\pi < \alpha, X_k(\omega) > - \frac{1}{q_N} \sum_{k=1}^N \mathbb{E}\ \exp\ 2i\pi < \alpha, X_k > \right|.$$

But χ is a positive random variable which is integrable because of Theorem 1. And it is obviously independent of the choice of the dynamical system.

Hence, as a corollary of Theorem 1, we obtain the following result which studies the asymptotic behavior of the ergodic means introduced in (1), within the framework of multidimensional dynamical systems, that is to say that the transform T is no longer a $\mathbb{Z}$-action but a $\mathbb{Z}^d$-action, $d \geq 1$.

Corollary 1 *Let $\mathcal{N} \subset \mathbb{N}$. For all m-dependent process $\{X_k, k \in \mathbb{N}\}$ with values in $\mathbb{N}^d$, and all normalization q_N satisfying the conditions (2) and (3) of Theorem 1, there exists a measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, there exists a finite positive constant $C = C(\omega, m)$ such that for all measurable dynamical system $(Y, \mathcal{A}, \mu, T)$, T being a $\mathbb{Z}^d$-action, for all $f \in L^2(\mu)$, we have*

$$\forall N \in \mathcal{N}, \quad \| \frac{1}{N} \sum_{k=1}^{N} f \circ T^{X_k(\omega)} - \frac{1}{N} \sum_{k=1}^{N} \mathbb{E} f \circ T^{X_k} \|_{L^2(\mu)} \leq C \| f \|_{L^2(\mu)} \frac{q_N}{N}.$$

This property is actually satisfied, more generally, for all probability space $(Y, \mathcal{A}, \mu)$ and all $L^2(\mu)$-contraction T.

Let us specify the notion of $\mathbb{Z}^d$-action. Suppose that $\{T_j : j \in \mathbb{Z}^d\}$ is a collection of invertible transformations of the measurable space $(Y, \mathcal{A}, \mu)$.

We say that $T := \{T_j : j \in \mathbb{Z}^d\}$ is a nonsingular action of $\mathbb{Z}^d$ on $(Y, \mathcal{A}, \mu)$ if

(i) $\exists X \in \mathcal{A}$ such that $\mu(Y - X) = 0$ and $T_j : X \to X$ $\forall j \in \mathbb{Z}^d$,

(ii) $(i, y) \to T_j(y)$ is measurable $(\mathbb{Z}^d, X \to X)$,

(iii) $T_i \circ T_j(y) = T_{i+j}(y)$ $\forall y \in X$, $i, j \in \mathbb{Z}^d$. Moreover, a nonsingular action of $\mathbb{Z}^d$ is called measure preserving if $\mu \circ T_j = \mu$, $\forall j \in \mathbb{Z}^d$. We say that the quadruplet $(Y, \mathcal{A}, \mu, T)$ a dynamical system. Thus an automorphism naturally induces a unitary operator on $L^2(\mu)$. If $S_1, S_2, \cdots, S_d$ are d commuting automorphisms on Y, and if $T = S_1 \circ S_2 \circ \cdots \circ S_d$, a typical example of $\mathbb{Z}^d$-actions is obtained by considering for any $j = (j_1, \cdots, j_d) \in \mathbb{Z}^d$, $T^j = S_1^{j_1} \circ S_2^{j_2} \circ \cdots \circ S_d^{j_d}$. We introduce here several approaches to obtain pointwise ergodic theorems for random processes. A first way consists in transferring the problem from Y to $\mathbb{Z}^d$ (replacing T by the shift), and to show maximal inequalities.

Let us recall that a $\mathbb{Z}^d$-valued sequence $(u_n)_{n\in\mathbb{N}}$ is said to be L^2-*good* if and only if, for all dynamical system $(Y, \mathcal{A}, \mu, T)$ and for all $f \in L^2(\mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f \circ T^{u_k} \text{ exists } \mu\text{-almost everywhere.}$$

More generally, we shall say that a sequence is L^2-*good for isometries* and L^2-*good for contractions*, if the mean above converges also for, respectively, all isometry and all contraction T.

In order to prove that a sequence is L^2-good, it is often convenient to show, in a first time, that there exists a positive constant C such that, for all $\rho > 1$,

$$\forall v \in l^2(\mathbb{Z}^d), \quad \left\| \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^{N} v(. + u_k) \right| \right\|_{l^2(\mathbb{Z}^d)} \leq C \| v \|_{l^2(\mathbb{Z}^d)}, \tag{6}$$

where

$$\mathcal{N}_\rho := \{[\rho^k],\, k \in \mathbb{N}\}, \tag{7}$$

and, using Definition 1 below, $\sum_{j=1}^{J} \| \tilde{M}_j^\rho v \|_{l^2(\mathbb{Z}^d)} / (J \| v \|_{l^2(\mathbb{Z}^d)})$ tends to 0, when J tends to ∞, at a speed which depends only on ρ.

Definition 1 *For a sequence of integers $(N_j)_{j\geq 1}$ such that $N_{j+1} \geq 2 N_j$, define*

$$\forall v \in l^2(\mathbb{Z}^d), \quad \tilde{M}_j^\rho v(n) := \sup_{N \in \mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^{N} v(n + u_k) - \frac{1}{N_j} \sum_{k=1}^{N_j} v(n + u_k) \right|, \tag{8}$$

where $\mathcal{N}_\rho^j := \{[\rho^k],\, k \in \mathbb{N},\, N_j \leq [\rho^k] < N_{j+1}\}$.

We shall therefore restrict ourselves to the sequences which satisfy these two properties. We shall call these sequences *good for maximal and variational inequalities* or *MVI-good*. These properties are just known to be sufficient conditions for a sequence to be L^2-good (the first one is actually also necessary).

Examples: For instance when $d = 1$, the sequences studied in [2],[18] and [20], $u_N = N$, $u_N = p_N$, the N^{th} primes number, and $u_N = [P(N)]$, the integer part of a polynomial $P(N)$ with real coefficients, are MVI-good sequences.

When $d \geq 2$, we can also choose $(u_N) \in \mathbb{N}^d$ as d polynomials with integer coefficients ([18] th. 6.2, [2]).

Our purpose, here, is to study the behavior of MVI-good sequences, when they are perturbed by a multiplicative, independent, identically distributed, random variable. More precisely, we show, the following.

Theorem 2 *Let $\{\theta_k : k \in \mathbb{N}\}$ and $\{\tilde{\theta}_k : k \in \mathbb{N}\}$ be two independant families of $\mathbb{N}^d$-valued i.i.d. random variables such that $\mathbb{E} \mid \prod_{i=1}^d \theta_0^{(i)} \mid^{1/2} < \infty$ and $\mathbb{E} \mid \prod_{i=1}^d \tilde{\theta}_0^{(i)} \mid^{\delta} < \infty$ for some $\delta > 0$. Let $(u_k)_{k\in\mathbb{N}}$ be a $\mathbb{N}^d$-valued MVI-good sequence such that $||u_k|| = O(2^{k^\beta})$ for some $\beta \in (0,1)$. Then there exists an absolute measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, for all measurable dynamical system $(Y, \mathcal{A}, \mu, T)$, for all $f \in L^2(\mu)$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f \circ T^{\theta_k(\omega)\, u_k + \tilde{\theta}_k(\omega)} \text{ exists } \mu\text{-almost everywhere.}$$

The result remains true when T is a non-negative contraction.

A second approach is directly located on the studied dynamical system using the spectral lemma. According to the general form of the process $\{X_k,\, k \in \mathbb{N}\}$, it is sometimes more convenient to use the first method or the second one. For instance, we shall prove the following result.

Theorem 3 *Let $(\Omega, \mathcal{B}, P)$ and $(\Omega', \mathcal{B}', P')$ be two independent probability spaces. Let S_k be a random walk generated by a sequence of non-centered i.i.d. random variables which have a moment of order two on $(\Omega', \mathcal{B}', P')$. Fix the walk and consider a sequence of integer-valued independent random variables $\{X_k,\, k \in \mathbb{N}\}$ which are defined on $(\Omega, \mathcal{B}, P)$.*

Moreover, assume that the law of the X_k's is generated by the convolution of a given law, that is to say, there exists an integrable random variable Y such that

$$\forall k \geq 1, \quad d\mathbb{P}_{X_k} = d\mathbb{P}_Y^{*(S_k)}. \tag{9}$$

Then there exists a measurable set $\Omega_0 \subset \Omega\times\Omega'$, with $P\times P'(\Omega_0) = 1$, such that, for all fixed $(\omega, \omega') \in \Omega_0$, we have:

For all measurable dynamical system $(X, \mathcal{A}, \mu, T)$, and for all $f \in L^2(\mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f \circ T^{X_k(\omega)} = \mathbb{E}(f|\mathcal{F}),\ \mu\text{-a.e.}$$

where $\mathcal{F}$ denotes the σ-field generated by T-invariant functions of $L^2(\mu)$.

The theorem remains true when T is a non-negative contraction.

Let us explain the meaning of the equation (9). When S_k is fixed, the process $\{X_k,\, k \in \mathbb{N}\}$ is independent and the law of each X_k is the same as $Y_1 + Y_2 + \ldots + Y_{S_k}$, where the Y_i's are i.i.d. and have the same law as Y. For instance, if Y is a Poisson law of parameter 1, X_k is a Poisson law of parameter S_k. Whereas, on $\Omega \times \Omega'$, the process $\{X_k,\, k \in \mathbb{N}\}$ is not independent.

We shall show more precisely that the oscillation function $O_{L^2(\mu)}(f \circ T^{X_k}, (N_j), \mathcal{N}_\rho)$ (see Section 6) associated with the sequence of ergodic means above is bounded by $K(\omega, \omega')||f||_{L^2(\mu)}$ where $P \times P'(K(\omega, \omega') < \infty) = 1$.

Here is another result that we shall prove using a similar technique.

Theorem 4 *Let $(\Omega, \mathcal{B}, P)$ be a complete probability space on which is defined a sequence of integer-valued independent random variables $\{X_k,\, k \in \mathbb{N}\}$. Moreover, assume that the law of the X_k's is generated by the convolution of a given law, that is to say, there exists an integrable random variable Y such that*

$$\forall k \geq 1, \quad d\mathbb{P}_{X_k} = d\mathbb{P}_Y^{*(u_k)},$$

where (u_k) is a sequence of integers that is assume to be MVI-good and such that there exists $\beta \in (0,1)$ with $u_k = O(2^{k^\beta})$. Then there exists a measurable set Ω_0, with $P(\Omega_0) = 1$, such that, for all fixed $\omega \in \Omega_0$, we have:

For all measurable dynamical system $(X, \mathcal{A}, \mu, T)$, and for all $f \in L^2(\mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f \circ T^{X_k(\omega)} \text{ exists } \mu\text{-a.e.}$$

The theorem remains true when T is a non-negative contraction.

The approach we shall use to prove the preceding results (in particular, Gaussian tools) can also be used in the study of convergence of ergodic means which are modulated by random weights. More precisely, we shall highlight the regularizing effect of the random weights generated by centered i.i.d. random variables that have a moment of order 2, in the pointwise ergodic theorem, for any subsequence.

Theorem 5 *Let $(\Omega, \mathcal{B}, P)$ be a complete probability space on which is defined a sequence of centered independent and identically distributed random variables $\{X_k,\, k \in \mathbb{N}\}$ that satisfies $\mathbb{E}|X_1|^2 < \infty$. Let $(u_k,\, k \geq 1)$ be an strictly increasing sequence of integer such that there exists $\beta \in (0,1)$ with $u_k = O(2^{k^\beta})$. Then there exists a set Ω_0, $\mathcal{B}$-measurable, with $P(\Omega_0) = 1$, such that, for all $\omega \in \Omega_0$, for all measurable dynamical system $(X, \mathcal{A}, \mu, T)$*

$$\forall f \in L^2(\mu), \quad \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} X_k(\omega)\, f \circ T^{u_k} = 0 \ \mu\text{-a.e.}$$

The theorem remains however true when T is a L^1-L^∞-contraction.

Remark that, when the sequence (u_k) is universally bad (see [14] for the definition), one can find a function $f \in L^2(\mu)$ which satisfies the property of strong sweeping out, that is to say,

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(T^{u_k}x) = 1, x\text{-}\mu\text{-a.e.}$$

and

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(T^{u_k}x) = 0, x\text{-}\mu\text{-a.e.}$$

In that extreme case (and if $u_k = O(2^{k^\beta})$), we know that, for almost every realization of universal sequence of centered random variables (X_k) with a moment of order 2, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} X_k f(T^{u_k}x) = 0, \ x\text{-}\mu\text{-a.e.}$$

A priori le preuve que nous donnons ne permet pas d'affirmer que le resultat est encore vrai pour toute contraction positive de L^2.

2 Gaussian Tools and proof of Theorem 1

2.1 technical lemmas

Let us present now the Gaussian tools proved by X.Fernique that we shall use in the sequel (see [15] for more details).

Proposition 1 *Let* $\{G_k, k \geq 1\}$ *be a sequence of Gaussian vectors with values in a Banach space* $(B, \| \,.\, \|)$. *Then*

$$\mathbb{E} \sup_{k \in \mathbb{N}} \| G_k \| \leq K_1 \left\{ \sup_{k \in \mathbb{N}} \mathbb{E} \| G_k \| + \mathbb{E} \sup_{k \in \mathbb{N}} | \lambda_k \sigma_k | \right\},$$

where $\{\lambda_k, k \geq 1\}$ *is an iso-normal sequence,* K_1 *an absolute constant and for all* $k \in \mathbb{N}$,

$$\sigma_k = \sup_{f \in B', \|f\| \leq 1} \| < G_k, f > \|_{2,\Omega}\,.$$

Proposition 2 *Let* g *be a real-valued Gaussian random function which is stationary, separable and continuous in quadratic mean. Moreover,* m *denotes its associated spectral measure on* $\mathbb{R}^+$ *such that*

$$\mathbb{E}(| g(s) - g(t) |^2) = 2 \int (1 - cos\ 2\pi u(s-t))\ m(du).$$

Then

$$\mathbb{E} \sup_{\alpha \in [0,1]} g(\alpha) \leq K_2 \left\{ (\int min(u^2, 1)\ m(du))^{1/2} + \int (m(]\exp\ x^2, \infty[))^{1/2} dx \right\},$$

where K_2 *is an absolute constant.*

Proposition 3 [Slépian-Fernique] *Let* T *be a finite set with cardinal* n, X *and* Y *be two Gaussian vectors with values in* $\mathbb{R}^d$, d_X *and* d_Y *be the associated deviations (that is to say, for instance,* $d_Y(s,t) = \sqrt{\mathbb{E} | Y(s) - Y(t) |^2}$, *for* $(s,t) \in T \times T$*). Assume that*

$$\forall (s,t) \in T \times T, \quad d_Y(s,t) \leq d_X(s,t).$$

Then, we have also

$$\mathbb{E} \sup_{t \in T} Y(t) \leq \mathbb{E} \sup_{t \in T} X(t).$$

We give one more spectral estimation which associates the operator T at the point f, with its spectral measure on the torus of dimension d, that we identify, once for all, with $[0,1[^d$. First for $d = 1$.

Proposition 4 *Let* T *be a contraction on a Hilbert space* $\mathcal{H}$, *and let* $p(x)$ *be a polynomial defined on the unit circle* $D = \{x, x \in \mathbb{C}, | x | = 1\}$. *Then for all* f *in* $\mathcal{H}$, *there exists a positive Borel measure, called* μ_f, *that is bounded over* D *and such that we have*

$$\| p(T)\ f \|^2_{\mathcal{H}} \leq \int_D | p(x) |^2\ \mu_f(dx).$$

This result is a consequence of Bochner Theorem that U.Krengel extended to contractions on Hilbert spaces (see [10], Proposition 3.1,p.94) and a dilation argument introduced by Sz.Nagy and Foias. (changer)

Now let $\mathbb{T}^d = [0,1[^d$ denote the dual group of $\mathbb{Z}^d$ and recall that if $\{T^k, k \in \mathbb{Z}^d\}$ is a group of unitary operators acting on $L^2(\mu)$, then it is easy to see that for each $f \in L^2(\mu)$, if we put for $k \in \mathbb{Z}^d$: $\gamma(k) = \int T^k f.\bar{f} d\mu$, then γ is a positive definite function, hence one has by the Herglotz-Bochner-Weil Theorem the existence of a unique non-negative bounded measure μ_f on the Borel $\sigma-$algebra $B([0,1[^d)$, such that

$$\forall k \in \mathbb{Z}^d, \qquad \int T^k f.\bar{f} d\mu = \int_{[0,1[^d} \exp 2i\pi < k, \alpha > \mu_f(d\alpha)\,.$$

Moreover if $n \in \mathbb{Z}^d$ and $P(T) = \sum_{k\in\mathbb{Z}^d,\, 0\le k\le n} a_k T^k$, then one has the spectral equality

$$\forall f \in L^2(\mu), \qquad ||P(T)(f)||^2 = \int_{[0,1[^d} \left| \sum_{k\in\mathbb{Z}^d,\, 0\le k\le n} a_k \exp 2i\pi < k, \alpha > \right|^2 \mu_f(d\alpha)\,,$$

and by $k \le n$ for $k, n \in \mathbb{Z}^d$ we mean $k_j \le n_j$ for each $1 \le j \le d$ (see also W. Ambrose, Spectral resolution of groups of unitary operators, Duke. Math. Journal, t. 11, 1944, p. 589-595.

Here are two other technical lemmas that we shall need.

Lemma 1 *[15] Let $\{f_n, n \ge 1\}$ be a sequence of non-negative and measurable functions defined on a measured space $(X, \mathcal{A}, \mu)$ with $\mu(X) = 1$. Assume that for all fixed $\rho > 1$,*

$$\mu\left\{x : \lim_{N\in\mathcal{N}_\rho\to\infty} \frac{1}{N}\sum_{k=1}^N f_k(x) \text{ exists}\right\} = 1,$$

where $\mathcal{N}_\rho$ is defined in (7). Then for all $\rho > 1$, the limit is the same, let us call it L, and we have

$$\mu\left\{x : \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N f_k(x) = L(x)\right\} = 1.$$

Lemma 2 *[2][18] Let $\{f_n, n \ge 1\}$ be a sequence of measurable functions defined on a measured space $(X, \mathcal{A}, \mu)$ with $\| f_n \|_{L^2(\mu)} = 1$, $\forall n \ge 1$. Let $\rho > 1$ and for all sequence of integer $(N_j)_{j\ge1}$ with $N_{j+1} \ge 2N_j$, let*

$$M_j(x) := \sup_{N\in\mathcal{N}_\rho^j} \left| \frac{1}{N}\sum_{k=1}^N f_k(x) - \frac{1}{N_j}\sum_{l=1}^{N_j} f_l(x) \right|,$$

where $\mathcal{N}_\rho^j$ is defined in Definition 1. Assume that $\sum_{j=1}^J \| M_j \|_{L^2(\mu)} /J$ tends to 0, when N tends to ∞, at a speed which does not depend on $(N_j)_{j\ge1}$. Then

$$\mu\left\{x : \lim_{N\in\mathcal{N}_\rho\to\infty} \frac{1}{N}\sum_{k=1}^N f_k(x) \text{ exists}\right\} = 1.$$

2.2 Dilation theorem and oscillation functions

We shall also use the following tools. Let us start with a *dilation* theorem from Akcoglu-Sucheston ([10], p.192, Th.2.9).

Theorem 6 [Dilation theorem] *Let $T : L \to L$ a non-negative contraction defined on a L^p-space L ($1 \le p < \infty$). Then there exists another L^p-space $\hat{L}$ and a non-negative isometry $\hat{T} : \hat{L} \to \hat{L}$ such that, for all $n \in \mathbb{N}$, $DT^n = P\hat{T}^n D$, where $D : L \to \hat{L}$ is a non-negative isometric embedding from L to $\hat{L}$, and $P : \hat{L} \to \hat{L}$ is a non-negative contraction.*

Consider now a probability space $(X, \mathcal{A}, \mu)$ and a sequence $\{f_k,\ k \ge 1\}$ of elements of $L^2(\mu)$ that satisfy: $\forall k \ge 1$, $||f_k||_{L^2(\mu)} = 1$. For all $\rho > 1$, consider, at last, a family of partial indexes $\mathcal{N}_\rho = \{[\rho^k],\ k \ge 0\}$, and an increasing sequence of integers, let us say $\{N_j,\ j \ge 0\}$, that satisfies: $N_{j+1} \ge 2N_j$, for all $j \ge 2$.

We are interested in the evaluation of the oscillation function associated with the sequence (f_k) of elements in $L^2(\mu)$, along the sequence $\{N_j,\ j \ge 0\}$. This $\bar{\mathbb{R}}$-valued function is defined as follows:

$$O_{L^2(\mu)}((f_k), (N_j), \mathcal{N}_\rho) := \sqrt{\sum_{j\ge1} \left\| \sup_{N\in\mathcal{N}_\rho^j} \left| \frac{1}{N}\sum_{k=1}^N f_k - \frac{1}{N_j}\sum_{k=1}^{N_j} f_k \right| \right\|_{L^2(\mu)}^2}\,, \qquad (10)$$

where $\mathcal{N}_\rho^j$ is given in Definition 1. For convenience, we shall also use, sometimes, the notation

$$O_{L^2(\mu)}((T_k),f,(N_j),\mathcal{N}_\rho) := \sqrt{\sum_{j\geq 1}\left\|\sup_{N\in\mathcal{N}_\rho^j}\left|\frac{1}{N}\sum_{k=1}^{N} f\circ T_k - \frac{1}{N_j}\sum_{k=1}^{N_j} f\circ T_k\right|\right\|_{L^2(\mu)}^2},$$

or

$$O_{L^2(\mu)}(T,f,(N_j),\mathcal{N}_\rho) := \sqrt{\sum_{j\geq 1}\left\|\sup_{N\in\mathcal{N}_\rho^j}\left|\frac{1}{N}\sum_{k=1}^{N} f\circ T^{u_k} - \frac{1}{N_j}\sum_{k=1}^{N_j} f\circ T^{u_k}\right|\right\|_{L^2(\mu)}^2},$$

where the choice of the integer-valued sequence (u_k) will be given by the context.

Proposition 5 *Assume there exists a positive constant $K<\infty$ which does not depends on the sequence $\{N_j,\, j\geq 1\}$, such that*

$$O_{L^2(\mu)}((f_k),(N_j),\mathcal{N}_\rho)\leq K.$$

Then we have:

non-negativity: *If we assume that, for all $k\geq 1$, $f_k\geq 0$, then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f_k(x) \text{ exists } \mu-pp. \tag{11}$$

boundedness: *If we assume that* $\sup_{k\geq 1}||f_k||_\infty<\infty$*, then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f_k(x) \text{ exists } \mu-pp. \tag{12}$$

Proof. We use, in both cases, Lemma 1. In the non-negative case, the conclusion comes down when it is combined with Lemma 2. In the second case, one replaces the argument of Lemma 2 by a standard reasoning that is clearly set in [18], p.719-04. ◇

2.3 Proof of Theorem 1

The data of the problem are the followings: Let $(\Omega,\mathcal{B},P)$ be a complete probability space on which is defined a random function $\{X_k, k\in\mathbb{N}\}$ with values in $\mathbb{R}^d$, $d\geq 1$. In order to set the results in the most general framework, assume that this function is m-dependent, that is to say, by definition,

Definition 2 *A random process $\{X_k,k\in\mathbb{N}\}$ is m-dependent if and only if, for any integer $k\geq 1$, X_{k+m} is independent of $\{X_k,X_{k-1},\cdots,X_0\}$.*

Examples: A sequence of independent random variables $\{X_k,k\in\mathbb{N}\}$ is by definition 1-dependent. On the other hand, a random walk $\{S_k=\sum_{j=0}^k X_j, k\in\mathbb{N}\}$ where $\{X_k,k\in\mathbb{N}\}$ is a sequence of independent and identically distributed random variables, is ∞-dependent if we fit the preceding definition in consequence.

We are going to estimate the asymptotic behavior of the quantity

$$\sup_{\alpha\in[0,1]^d}\left|\sum_{k=1}^{N}\exp\ 2i\pi<\alpha,X_k>-\mathbb{E}\ \exp\ 2i\pi<\alpha,X_k>\right|, \tag{13}$$

uniformly in α when N tends to infinity. That is the essential point of this work that will permit to separate the conditions to impose, on one hand, on the random process, and on the other hand, on the dynamical system that is likely to model a concrete situation, as for instance, the position of an energetic particle in the space.

-Step 1: This step consists in transferring the problem to a problem of regularity of Gaussian random functions'trajectories.

Let us take an independent copy of $X = (X_k)_{k\geq 0}$ that we denote $X' = (X'_k)_{k\geq 0}$. Usual convexity properties shows that, to prove (4), we just need to prove

$$\mathbb{E}_{X,X'} \sup_{N\in\mathcal{N}} \sup_{\alpha\in[0,1]^d} \left| \frac{1}{q_N} \sum_{k=1}^{N} \exp\ 2i\pi < \alpha, X_k > - \exp\ 2i\pi < \alpha, X'_k > \right| < \infty.$$

Symetrization of the problem: Let us consider the following family of random functions with continuous trajectories,

$$\{f_k(\alpha) = \exp\ 2i\pi < \alpha, X_k > - \exp\ 2i\pi < \alpha, X'_k >, \alpha \in [0,1]^d, k \geq 1\}.$$

The hypothesis of m-dependence on the process $\{X_k, k \geq 1\}$ suggests to make a partition of $\mathbb{N}$, the set of positive integers, as follows:

$$\mathbb{N} = I_1 \cup I_2 \cup \cdots \cup I_m,$$

where, for all integers $l \leq m$, $I_l = \{l + rm, r \in \mathbb{N}\}$. Our goal is to construct a finite family (with m elements) of symmetric random functions associated with the m arithmetic progressions.

For all integers $l \geq m$, we denote the random function

$$f^l = \{f_k, k \in I_l\}.$$

Note that, by construction, every f^l forms a symmetric family of random functions, that is to say that their law is not modified by change of sign. More precisely, let us denote $\{\epsilon_k, k \geq 1\}$ a sequence of Rademacher, independent, random variables (which takes the values $+1$ and -1 with probability $1/2$) that we assume to be independent of the random variables X and X'.

Hence, we have

$$\sum_{k=1}^{N} f_k(\alpha) = \sum_{l=1}^{m} \sum_{k\in I^l, k\leq N} f_k(\alpha),$$

and so

$$\mathbb{E}_{X,X'} \sup_{N\in\mathcal{N}} \sup_{\alpha\in[0,1]^d} \left| \frac{1}{q_N} \sum_{k=1}^{N} f_k(\alpha) \right| \leq \sum_{l=1}^{m} \mathbb{E}_{X,X'} \sup_{N\geq 1} \sup_{\alpha\in[0,1]^d} \left| \frac{1}{q_N} \sum_{k\in I^l, k\leq N} f_k(\alpha) \right|.$$

Let us fix now an integer $l \leq m$ and study

$$\mathbb{E}_{X,X'} \sup_{N\in\mathcal{N}} \sup_{\alpha\in[0,1]^d} \left| \frac{1}{q_N} \sum_{k\in I^l, k\leq N} f_k(\alpha) \right|. \tag{14}$$

We know that $\{f_k, k \in I^l\}$ and $\{\epsilon_k f_k, k \in I^l\}$ have the same law. Thus (14) can be written with a larger space of integration,

$$\mathbb{E}_{X,X',\epsilon} \sup_{N\in\mathcal{N}} \sup_{\alpha\in[0,1]^d} \left| \frac{1}{q_N} \sum_{k\in I^l, k\leq N} \epsilon_k f_k(\alpha) \right| \leq$$

$$2\ \mathbb{E}_{X,\epsilon} \sup_{N\in\mathcal{N}} \sup_{\alpha\in[0,1]^d} \left| \frac{1}{q_N} \sum_{k\in I^l, k\leq N} \epsilon_k \exp\ 2i\pi < \alpha, X_k > \right|. \tag{15}$$

We shall use a tool which is recurrent in the study of Gaussian processes: the principle of contraction. This tool is based on a very simple idea which allows us to replace the change of sign in (15) by a sequence of centered and reduced, normal, random variables. For more details, see [15] page 169.

Hence, to show that the right term of (15) is finite, we just need to prove

$$\mathbb{E}_{X,g,g'} \sup_{N\in\mathcal{N}} \sup_{\alpha\in[0,1]^d} \left| \frac{1}{q_N} \sum_{k\in I^l, k\leq N} g_k \cos\ 2\pi < \alpha, X_k > + g'_k \sin\ 2\pi < \alpha, X_k > \right| < \infty, \tag{16}$$

where $\{g_k, k \geq 1\}$ and $\{g'_k, k \geq 1\}$ are two sequences of independent random variables of law $\mathcal{N}(0,1)$ which are independent of X, ϵ and ϵ'.

The problem is then reduced to the study of the regularity of Gaussian random functions'trajectories if we fix the variable of integration X. This concludes the first step of the proof.

-Step 2: Use of Gaussian tools.

Let us start by denoting, conditionally to X, the random Gaussian functions

$$G_N(\alpha) = \frac{1}{q_N} \sum_{k \in I^l, k \leq N} g_k \cos\ 2\pi < \alpha, X_k > + g'_k \sin\ 2\pi < \alpha, X_k > .$$

Because of Proposition 5, we can bound (16) from above by

$$\mathbb{E} \sup_{N \in \mathcal{N}} \sup_{\alpha \in [0,1]^d} \mid G_N(\alpha) \mid \leq K_1 \mathbb{E}_X \left(\sup_{N \geq 1} \mathbb{E}_{g,g'} \sup_{\alpha \in [0,1]^d} \mid G_N(\alpha) \mid + \mathbb{E}_\lambda \sup_{N \geq 1} \mid \lambda_N \sigma_N \mid \right),$$

where $\{\lambda_N, N \geq 1\}$ is an iso-normal sequence, that is to say, a sequence of independent and identically distributed, random variables with centered and reduced, Gaussian law and

$$\sigma_N \leq \sup_{\alpha \in [0,1]^d} \parallel G_N(\alpha) \parallel_{2,g,g'} .$$

This estimation of σ_N permits to obtain easily

$$\sigma_N \leq \frac{1}{q_N} \sqrt{card\{k \in I^l, k \leq N\}}.$$

or

$$\sigma_N \leq \frac{\sqrt{N}}{q_N}.$$

But the hypothesis (2) tells us that $\sigma_N = o(1/\sqrt{\log N})$. Thus, using results on strong integration of Gaussian vectors, we obtain

$$\mathbb{E}_\lambda \sup_{N \in \mathcal{N}} \mid \lambda_N \sigma_N \mid < \infty,$$

and finally,

$$\mathbb{E}_X \mathbb{E}_\lambda \sup_{N \in \mathcal{N}} \mid \lambda_N \sigma_N \mid < \infty.$$

Hence to show (16), we just need to prove

$$\mathbb{E}_X \sup_{N \in \mathcal{N}} \mathbb{E}_{g,g'} \sup_{\alpha \in [0,1]^d} \mid G_N(\alpha) \mid < \infty. \tag{17}$$

In the sequel, since X is fixed, we shall fix also the integer N. We are going to compare the random function $G(\alpha) = G_N(\alpha)$ with every one of its 1-dimensional margins in order to use Proposition 7. The random function G represents a stationary Gaussian random function which is continuous in probability and thus, possess a modification with continuous trajectory that will be also denoted G. On the space $[0,1]^d$, G induces an Hilbert distance defined by

$$\forall (s,t) \in [0,1]^d \times [0,1]^d, \quad d_G(s,t) = \sqrt{\mathbb{E} \mid G(s) - G(t) \mid^2}.$$

For a positive integer $j \leq d$, let us consider the family of stationary, Gaussian, random functions which are continuous in probability

$$\forall \alpha_j \in [0,1], \quad G^j(\alpha_j) = \frac{1}{q_N} \sum_{k \in I^l, k \leq N} g_{k,j} \cos\ 2\pi \alpha_j X_k^{(j)} + g'_{k,j} \sin\ 2\pi \alpha_j X_k^{(j)},$$

where $\{g_{k,j}, k \geq 1\}$ and $\{g'_{k,j}, k \geq 1\}$ are families of sequences of centered and reduced, Gaussian, random variables which are independent for $j \leq d$ and $l \leq m$.

For all t_j and s_j in $[0,1]$, denote

$$d_{G^j}(s_j,t_j)=\sqrt{\mathbb{E}\mid G^j(s_j)-G^j(t_j)\mid^2},$$

the Hilbert distance on $[0,1]$, induced by G^j. In this context, remark that we have

$$\forall t=(t_1,\cdots,t_d)\in[0,1]^d, \forall s=(s_1,\cdots,s_d)\in[0,1]^d,\ \ d_G(s,t)\le\sqrt{d}\sqrt{\sum_{j=1}^d d^2_{G^j}(s_j,t_j)}.$$

Using Slépian-Fernique's Theorem (see Proposition 3), we deduce from the preceding estimation

$$\mathbb{E}_{g,g'}\sup_{\alpha\in[0,1]^d}G_N(\alpha)\le\sqrt{d}\sum_{j=1}^d\mathbb{E}_{g,g'}\sup_{\alpha_j\in[0,1]}G^j_N(\alpha_j).$$

For all $j\le d$, we associate the random function G^j with its spectral measure on $\mathbb{R}^+$ defined by

$$\forall 1\le j\le d,\qquad m^{(j)}=\frac{1}{q_N^2}\sum_{k\in I^l,k\le N}\delta_{|X_k^{(j)}|},$$

where δ_u is the Dirac measure at the point u.

Because of Proposition 2, we obtain that (17) will be achieved as soon as the two families of following conditions will be satisfied

$$\forall j\le d,\quad \mathbb{E}\sup_{N\in\mathcal{N}}\left[\frac{1}{q_N}\right]\sqrt{\sum_{k\in I^l,k\le N}\min[(X_k^{(j)})^2,1]}<\infty. \tag{18}$$

$$\forall j\le d,\quad \mathbb{E}\sup_{N\in\mathcal{N}}\left[\frac{1}{q_N}\right]\int_0^\infty\sqrt{\sum_{k\in I^l,k\le N}I_{\{|X_k^{(j)}|>e^{x^2}\}}}\,dx<\infty. \tag{19}$$

Note that (18) can be reduced to $\sup_{N\in\mathcal{N}}(\sqrt{N}/q_N)<\infty$. But this is an immediate consequence of the hypothesis (2). Finally, the condition (19) can be easily reduced to the condition (3) by using the non-increasing condition on $(\sqrt{N}/q_N)$. This ends the proof of Theorem 1.

◇

2.4 Another result

At last, we shall also need the following proposition relating to some random trigonometric polynomials and that we prove with the help of the Gaussian techniques clearly set now.

Proposition 6 *Let $(\Omega,\mathcal{B},P)$ be a probability space on which is defined a sequence of centered, independent and identically distributed random variables $(\theta_k,\ k\ge 1)$ which satisfies $\mathbb{E}|\theta_k|^2<\infty$. Let $(u_k,\ k\ge 1)$ be an increasing sequence of reals such that there exists $c_1>0$ and $c_2>0$ with $u_N\ge c_1N^{c_2}$.*

Then, we have

$$\mathbb{E}\sup_{N\ge 1}\sup_{\alpha\in[0,1]}\left|\frac{1}{\sqrt{N\log u_N}}\sum_{k=1}^N\theta_k\exp(2i\pi\alpha u_k)\right|<\infty.$$

For the proof of this result, the reader is referred to [17].

3 First approach: an application of the transference method

3.1 A transference theorem

The aim of the following theorem is to explore a transference principle from a model with one deterministic orbit ($\mathbb{Z}^d$, the shift S) to any random dynamical system.

Definition 3 *For all real $\rho > 1$, define a partial index*

$$\mathcal{N}_\rho = \{[\rho^k],\ k \in \mathbb{N}\},$$

and for all sequence of integers $(N_j)_{j\geq 1}$ with $N_{j+1} \geq 2\,N_j$,

$$\mathcal{N}_\rho^j = \{[\rho^k],\ k \in \mathbb{N},\, N_j \leq [\rho^k] < N_{j+1}\}.$$

For all $v \in l^2(\mathbb{Z}^d)$, define also

$$M^\rho v(n) = \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}\, v(n + X_k) \right|, \tag{20}$$

and

$$M_j^\rho v(n) = \sup_{N \in \mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}\, v(n + X_k) - \frac{1}{N_j} \sum_{k=1}^{N_j} \mathbb{E}\, v(n + X_k) \right|. \tag{21}$$

Theorem 7 *Let be a $\mathbb{N}^d$-valued random process $\{X_k,\ k \in \mathbb{N}\}$ which satisfies*

$$\mathbb{E} \parallel X_k \parallel^\delta = O(2^{k^\gamma}),$$

for some $\delta > 0$ and $\gamma < 1$, and assume there exist a constant C and a number $A(J)$ independent of $(N_j)_{j\geq 1}$ such that, for all $v \in l^2(\mathbb{Z}^d)$,

$$\parallel M^\rho v \parallel_{l^2(\mathbb{Z}^d)} \leq C \parallel v \parallel_{l^2(\mathbb{Z}^d)}, \tag{22}$$

$$\sum_{j=1}^{J} \parallel M_j^\rho v \parallel_{l^2(\mathbb{Z}^d)} \leq A(J) \parallel v \parallel_{l^2(\mathbb{Z}^d)}, \tag{23}$$

and $A(J)/J$ tends to 0 when $J \to \infty$ at a speed which depends only on ρ. Then there exists an absolute, measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, for all measurable dynamical system $(Y, \mathcal{A}, \mu, T)$, T being a $\mathbb{Z}^d$-action, for all $f \in L^2(\mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f \circ T^{X_k(\omega)} \text{ exists } \mu\text{-almost everywhere.}$$

The theorem remains true when T is a non-negative contraction.

This result gives us deterministic criteria for a particular dynamical system $\mathbb{Z}^d$ with the shift, which lead to general properties of pointwise convergence of random ergodic means within a universal framework.

In order to prove Theorem 7, we need the following lemmas.

Lemma 3 *Let $\rho > 1$ and let $\{X_k,\ k \in \mathbb{N}\}$ be a $\mathbb{N}^d$-valued random process such that*

$$\forall v \in l^2(\mathbb{Z}^d), \quad \parallel M^\rho v \parallel_{l^2(\mathbb{Z}^d)} \leq C \parallel v \parallel_{l^2(\mathbb{Z}^d)},$$

where $M^\rho v$ is defined by (20). Then, for all dynamical system $(Y, \mathcal{A}, \mu, T)$, we have

$$\forall f \in L^2(\mu), \quad \left\| \sup_{N \in \mathcal{N}_\rho} \left| \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}(f \circ T^{X_k}) \right| \right\|_{L^2(\mu)} \leq C \parallel f \parallel_{L^2(\mu)} .$$

The result remains true when T is a non-negative isometry.

Lemma 4 *Let $\rho > 1$ and let $(N_j)_{j\geq 1}$ be a sequence of integers such that $N_{j+1} \geq 2\,N_j$ and let M_j^ρ be defined by (21). Assume there exist, for all fixed J, a number $A(J)$ independent of v and $(N_j)_{j\geq 1}$ such that*

$$\sum_{j=1}^{J} \| M_j^\rho v \|_{l^2(\mathbb{Z}^d)} \leq A(J) \; \| v \|_{l^2(\mathbb{Z}^d)} \; .$$

Then , for all dynamical system $(Y, \mathcal{A}, \mu, T)$, we have, $\forall f \in L^2(\mu)$,

$$\sum_{j=1}^{J} \left\| \sup_{N\in\mathcal{N}_\rho^j} \left| \frac{1}{N}\sum_{k=1}^{N} \mathbb{E}(f\circ T^{X_k}) - \frac{1}{N_j}\sum_{k=1}^{N_j} \mathbb{E}(f\circ T^{X_k}) \right| \right\|_{L^2(\mu)} \leq A(J) \; \| F \|_{L^2(\mu)} \; .$$

The result remains true when T is a non-negative isometry.

Lemmas 3 and 4 are generalization of transference principles based on a model with one orbit [2] (the only novelty here is in the presence of the expectation $\mathbb{E}$). We just give the proof of the first lemma, the second one is obtained from the original result by the same modifications.

Proof. First, let us remark that we just need to prove Lemma 3 for non negative functions. Fix $m \in \mathbb{N}$ and take an integer $J > m$. Let

$$\varphi_x(n) = \begin{cases} T^n \circ f(x) & \text{if } n \in \{0, \cdots, J\}^d \\ 0 & \text{otherwise} \end{cases}$$

¿From the hypothesis of the lemma, we have

$$\left\| \sup_{N\in\mathcal{N}_\rho} \left(\frac{1}{N}\sum_{k=1}^{N} \mathbb{E}\,\varphi_x(.+X_k) \right) \right\|_{l^2(\mathbb{Z}^d)} \leq C \; \| \varphi_x \|_{l^2(\mathbb{Z}^d)},$$

and since φ_x is a non negative function, we have also

$$\left\| \sup_{N\in\mathcal{N}_\rho} \left(\frac{1}{N}\sum_{k=1}^{N} \mathbb{E}\left(\varphi_x(.+X_k)\, I_{\{\omega: X_k^{(j)}\leq m,\ \forall j=1,\cdots,d\}} \right) \right) \right\|_{l^2(\mathbb{Z}^d)} \leq C \; \| \varphi_x \|_{l^2(\mathbb{Z}^d)} \; .$$

Hence, since $\varphi_x(n+X_k)\, I_{\{\omega: X_k^{(j)}\leq m,\ \forall j=1,\cdots,d\}} = f(T^{n+X_k}x)\, I_{\{\omega: X_k^{(j)}\leq m,\ \forall j=1,\cdots,d\}}$ for $n \in \{0, \cdots, J-m\}^d$,

$$\sum_{n_1=0}^{J-m} \cdots \sum_{n_d=0}^{J-m} \sup_{N\in\mathcal{N}_\rho} \left(\frac{1}{N}\sum_{k=1}^{N} \mathbb{E}\left(T^{n+X_k}\circ f(x)\, I_{\{\omega: X_k^{(j)}\leq m,\ \forall j=1,\cdots,d\}} \right) \right)^2 \leq C^2 \sum_{k_1=0}^{J} \cdots \sum_{k_d=0}^{J} (T^k \circ f(x))^2.$$

But T is measure preserving and, by integrating the last inequality, we obtain

$$(J-m)^d \left\| \sup_{N\in\mathcal{N}_\rho} \left(\frac{1}{N}\sum_{k=1}^{N} \mathbb{E}\left(T^{X_k}\circ f\, I_{\{\omega: X_k^{(j)}\leq m,\ \forall j=1,\cdots,d\}} \right) \right) \right\|_{L^2(\mu)}^2 \leq C^2 J^d \; \| f \|_{L^2(\mu)}^2,$$

that is to say

$$\left\| \sup_{N\in\mathcal{N}_\rho} \left(\frac{1}{N}\sum_{k=1}^{N} \mathbb{E}\left(T^{X_k}\circ f\, I_{\{\omega: X_k^{(j)}\leq m,\ \forall j=1,\cdots,d\}} \right) \right) \right\|_{L^2(\mu)} \leq C \; \| f \|_{L^2(\mu)} \left(\frac{J}{J-m} \right)^{\frac{d}{2}},$$

and making J tend to infinity,

$$\left\| \sup_{N\in\mathcal{N}_\rho} \left(\frac{1}{N}\sum_{k=1}^{N} \mathbb{E}\left(T^{X_k}\circ f\, I_{\{\omega: X_k^{(j)}\leq m,\ \forall j=1,\cdots,d\}} \right) \right) \right\|_{L^2(\mu)} \leq C \; \| f \|_{L^2(\mu)} \; .$$

We just have now to make m tend to infinity. Since, for a fixed $\bar{N} \in \mathbb{N}$,

$$\sup_{N\in\mathcal{N}_\rho,\, N\le\bar{N}} \left(\frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\left(f(T^{X_k}x)\, I_{\{\omega: X_k^{(j)}\le m,\ \forall j=1,\cdots,d\}}\right)\right)$$

tends to $\sup_{N\in\mathcal{N}_\rho,\, N\le\bar{N}}\left(\frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\left(f(T^{X_k}x)\right)\right)$ for μ-almost every x in X, using Lebesgue's monotonous convergence Theorem we have proved that

$$\left\|\sup_{N\in\mathcal{N}_\rho,\, N\le\bar{N}}\left(\frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\left(T^{X_k}\circ f\right)\right)\right\|_{L^2(\mu)} \le C\ \| f \|_{L^2(\mu)}\ .$$

Then making $\bar{N}$ tend to infinity and using

$$\left\|\sup_{N\in\mathcal{N}_\rho}\left|\frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\left(T^{X_k}\circ f\right)\right|\right\|_{L^2(\mu)} \le$$

$$\left\|\sup_{N\in\mathcal{N}_\rho}\left(\frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\left(T^{X_k}\circ f\right)\right)^{+}\right\|_{L^2(\mu)} + \left\|\sup_{N\in\mathcal{N}_\rho}\left(\frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\left(T^{X_k}\circ f\right)\right)^{-}\right\|_{L^2(\mu)},$$

we obtain the expected inequality. ◇

Using Corollary 1 together with Lemma 3, we obtain

Corollary 2 *Let be a $\mathbb{N}^d$-valued random process $\{X_k, k\in\mathbb{N}\}$ which satisfies*

$$\mathbb{E}(\| X_N \|^\delta) = O(2^{N^\gamma}),$$

for some $\delta > 0$ and $\gamma < 1$. Let $\rho > 1$ and assume there exists a constant C independent of $v \in l^2(\mathbb{Z}^d)$ such that

$$\| M^\rho v \|_{l^2(\mathbb{Z}^d)} \le C\ \| v \|_{l^2(\mathbb{Z}^d)},$$

then there exists an absolute, measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, there exists a finite, positive constant $C = C(\omega, m)$ such that for all measurable dynamical system $(Y, \mathcal{A}, \mu, T)$, T being a $\mathbb{Z}^d$-action, for all $f \in L^2(\mu)$, we have

$$\left\|\sup_{N\ge1}\left|\frac{1}{N}\sum_{k=1}^{N} f\circ T^{X_k(\omega)}\right|\right\|_{L^2(\mu)} \le C(\omega)\ \| f \|_{L^2(\mu)}\ .$$

The result remains true when T is a non-negative isometry.

Proof. First of all, let $\gamma' \in ((\gamma+1)/2, 1)$ and notice that $q_N := N^{\gamma'}$ satisfies the conditions of non-increasing, (2), (3) and $\sum_{N\in\mathcal{N}_\rho} q_N/N < \infty$. Indeed, the first and the last conditions are easily checked and, for (3), we proceed as follow.

Multiplying (3) by $\sqrt{\delta}$ and changing q_N in $N^{\gamma'}$, we have to bound

$$\forall 1\le i\le d \qquad \mathbb{E}_X \sup_{N\in N}\sqrt{\frac{\log^+(|\ X_N^{(i)}\ |^\delta)}{N^{2\gamma'-1}}},$$

from above. It is actually smaller than

$$\left(\mathbb{E}_X \sup_{N\in N}\sqrt{\frac{\log^+(|\ X_N^{(i)}\ |^\delta / 2^{N^{2\gamma'-1}})}{N^{2\gamma'-1}}} + \sup_{N\in N}\sqrt{\frac{\log(2^{N^{2\gamma'-1}})}{N^{2\gamma'-1}}}\right),$$

where the second term is bounded from above by 1 and the first term is bounded by

$$\sum_{N\in N} \sqrt{\frac{\mathbb{E}_X \log^+(\mid X_N^{(i)} \mid^\delta /2^{N^{2\gamma'-1}})}{N^{2\gamma'-1}}} \leq \sum_{N\in\mathcal{N}_\rho} \sqrt{\frac{\mathbb{E}_X(\mid X_N^{(i)} \mid^\delta)}{2^{N^{2\gamma'-1}} N^{2\gamma'-1}}}.$$

Using the hypothesis made on X_N, one proves that it is smaller than

$$\sum_{N\in N} \sqrt{\frac{2^{N^\gamma}}{2^{N^{2\gamma'-1}} N^{2\gamma'-1}}} < \infty.$$

Letting

$$A_N := \frac{1}{N}\sum_{k=1}^N f\circ T^{X_k(\omega)},$$

we have next,

$$\| \sup_{N\in\mathcal{N}_\rho} \mid A_N \mid \|_{L^2(\mu)} \quad \leq \quad \| \sup_{N\in\mathcal{N}_\rho} \mid A_N - \mathbb{E}A_N \mid \|_{L^2(\mu)} + C \parallel f \parallel_{L^2(\mu)}$$

And the non-negativity of T permits to change the supremum over $\mathcal{N}_\rho$ into the supremum over $\mathbb{N}$ (for $\rho^p \leq N \leq \rho^{p+1}$, we have $\mid A_N \mid \leq \rho \mid A_{\rho^{p+1}} \mid$).

◇

Corollary 2 means that the class of convergence of the random ergodic means above, for the almost everywhere convergence, is closed in $L^2(\mu)$.

The problem of pointwise convergence is thus linked to the possibility of making an approximation: find a class which is dense in $L^2(\mu)$ and for which we have almost-everywhere convergence. We have actually the following corollary of Corollary 1 and Lemma 4.

Corollary 3 *Let the process $\{X_k, k\in\mathbb{N}\}$ satisfies*

$$\mathbb{E}(\| X_N \|^\delta) = O(2^{N^\gamma}),$$

for some $\delta > 0$ and $\gamma < 1$. Let $(N_j)_{j\geq 1}$ be a sequence of integers such that $N_{j+1} \geq 2\,N_j$ and let M_j^ρ be defined by (21). Assume there exists, for all fixed J, a number $A(J)$ independent of v and $(N_j)_{j\geq 1}$ such that

$$\forall v\in l^2(\mathbb{Z}^d),\quad \sum_{j=1}^J \| M_j^\rho v \|_{l^2(\mathbb{Z}^d)} \leq A(J) \ \| v \|_{l^2(\mathbb{Z}^d)},$$

where $A(J)/J$ tends to 0 when $J\to\infty$ at a speed which depends only on $\rho > 1$, then we obtain:

There exists an absolute, measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega\in\Omega_0$, for all measurable dynamical system $(Y, \mathcal{A}, \mu, T)$, T being a $\mathbb{Z}^d$-action, for all $f\in L^\infty(\mu)$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N f\circ T^{X_k(\omega)} \text{ exists } \mu\text{-almost everywhere.}$$

The result remains true when T is a non-negative isometry.

Proof. Using the same notation as above, Lemmas 4 and 2 tells us that $\mathbb{E}A_N$ converges when $N\in\mathcal{N}_\rho$ and $N\to\infty$. But, from Corollary 1,

$$\| \sum_{N\in\mathcal{N}_\rho} (A_N - \mathbb{E}A_N) \|_{L^2(\mu)} \leq \sum_{N\in\mathcal{N}_\rho} \| A_N - \mathbb{E}A_N \|_{L^2(\mu)} < \infty.$$

Therefore

$$\sum_{N\in\mathcal{N}_\rho} (A_N - \mathbb{E}A_N) < \infty, \quad \mu - a.e.$$

which means that $A_N - \mathbb{E}A_N$ tends to 0, and A_N converges μ-almost everywhere, when $N\in\mathcal{N}_\rho$. And, since T is non-negative, we know, from Lemma 1, that the convergence still occurs when N describes $\mathbb{N}$. Finally, remark that the dilation theorem (in Section 2.2) gives us, also, the result for non-negative isometries. ◇

Corollary 2 together with Corollary 3 lead obviously to Theorem 7.

3.2 Application to perturbed MVI-good sequences (proof of Theorem 2)

Our purpose, here, is to study the behavior of MVI-good sequences when they are perturbed by a multiplicative, independent, identically distributed, random variable. First of all, we have

Lemma 5 *Let $(u_n)_{n\in\mathbb{N}}$ be $\mathbb{N}^d$-valued sequence which satisfies (6), then for all $m=(m_i)_{i=1}^d \in \mathbb{Z}^d$, we have*

$$\forall v \in l^2(\mathbb{Z}^d), \quad \left\|\sup_{N\in\mathcal{N}_\rho}\left|\frac{1}{N}\sum_{k=1}^N v(.+m\,u_k)\right|\right\|_{l^2(\mathbb{Z}^d)} \leq C \parallel v \parallel_{l^2(\mathbb{Z}^d)},$$

with the same constant C.

To prove Lemma 5, we just use an Euclidean division on $\mathbb{Z}^d$. Let

$$\begin{array}{rcl} S & = & \sum_{n\in\mathbb{Z}^d} \mid \sup_{N\in\mathcal{N}_\rho} \mid \frac{1}{N}\sum_{k=1}^N v(n+m\,u_k) \mid\mid^2 \\ & = & \sum_{l_1=1}^{m_1}\cdots\sum_{l_d=1}^{m_d}\sum_{n\in\mathbb{Z}^d} \mid \sup_{N\in\mathcal{N}_\rho} \mid \frac{1}{N}\sum_{k=1}^N v(m(n+u_k)+l) \mid^2 \end{array}$$

Applying the hypothesis of the lemma to $n\mapsto f(m\,n+l)$, we obtain

$$S \leq \sum_{l_1=1}^{m_1}\cdots\sum_{l_d=1}^{m_d} C^2 \sum_{n\in\mathbb{Z}^d} \mid v(m\,n+l)\mid^2 = C^2\sum_{n\in\mathbb{Z}^d} \mid v(n)\mid^2$$

◇

Corollary 4 *Let $\{\theta_N,\ N\in\mathbb{N}\}$ be a family of independent, identically distributed, $\mathbb{Z}^d$-valued random variables and $(u_n)_{n\in\mathbb{N}}$ a $\mathbb{Z}^d$-valued sequence which satisfies (6), then we have*

$$\forall v \in l^2(\mathbb{Z}^d), \quad \left\|\sup_{N\in\mathcal{N}_\rho}\left|\frac{1}{N}\sum_{k=1}^N \mathbb{E}\, v(.+\theta_k u_k)\right|\right\|_{l^2(\mathbb{Z}^d)} \leq C \parallel v \parallel_{l^2(\mathbb{Z}^d)}.$$

For $j\in\mathbb{Z}^d$, denote $P_j = P(\{\omega\in\Omega:\ \theta_0(\omega)=j\})$. Since the family $\{\theta_N,\ N\in\mathbb{N}\}$ is identically distributed, we have also $P_j = P(\{\omega\in\Omega:\ \theta_N(\omega)=j\})$. Hence

$$\begin{array}{rcl} \sup_{N\in\mathcal{N}_\rho} \mid \frac{1}{N}\sum_{k=1}^N \mathbb{E}\, v(n+\theta_k u_k)\mid & = & \sup_{N\in\mathcal{N}_\rho} \mid \frac{1}{N}\sum_{k=1}^N \sum_{j\in\mathbb{Z}^d} P_j v(n+j\,u_k)\mid \\ & \leq & \sum_{j\in\mathbb{Z}^d} P_j \sup_{N\in\mathcal{N}_\rho} \mid \frac{1}{N}\sum_{k=1}^N v(n+j\,u_k)\mid. \end{array}$$

Using the triangular inequality, we can bound the l^2-norm from above as follow

$$\left\|\sup_{N\in\mathcal{N}_\rho}\left|\frac{1}{N}\sum_{k=1}^N \mathbb{E}\, v(.+X_k)\right|\right\|_{l^2(\mathbb{Z}^d)} \leq \sum_{j\in\mathbb{Z}} P_j \left\|\sup_{N\in\mathcal{N}_\rho}\left|\frac{1}{N}\sum_{k=1}^N v(.+j\,u_k)\right|\right\|_{l^2(\mathbb{Z}^d)}.$$

And using Lemma 5, we obtain

$$\left\|\sup_{N\in\mathcal{N}_\rho}\left|\frac{1}{N}\sum_{k=1}^N \mathbb{E}\, v(.+X_k)\right|\right\|_{l^2(\mathbb{Z}^d)} \leq \sum_{j\in\mathbb{Z}} P_j C \parallel v \parallel_{l^2(\mathbb{Z}^d)} = C \parallel v \parallel_{l^2(\mathbb{Z}^d)}.$$

◇

We have found therefore a process $(X_N = \theta_N u_N)$ which satisfies (22). To apply Theorem 7, we are going to show that it satisfies also (23).

Recall that $\tilde{M}_j^\rho$ is defined in (8), and for $m=(m_i)_{i=1}^d\in\mathbb{Z}^d$, define

$$\forall v\in l^2(\mathbb{Z}^d), \quad \tilde{M}_j^{\rho,m} v(n) = \sup_{N\in\mathcal{N}_\rho^j}\left|\frac{1}{N}\sum_{k=1}^N v(n+m\,u_k) - \frac{1}{N_j}\sum_{k=1}^{N_j} v(n+m\,u_k)\right|.$$

Lemma 6 *Let $(u_k)_{k\in\mathbb{N}}$ be an $\mathbb{Z}^d$-valued sequence such that there exists, for all fixed J, a number $A(J)$ independent of v and $(N_j)_{j\geq 1}$ which satisfies*

$$\forall v \in l^2(\mathbb{Z}^d), \quad \sum_{j=1}^{J} \| \tilde{M}_j^{\rho} v \|_{l^2(\mathbb{Z}^d)} \leq A(J) \ \| v \|_{l^2(\mathbb{Z}^d)} \ .$$

Then, we have, for all $m \in \mathbb{Z}^d$,

$$\forall v \in l^2(\mathbb{Z}^d), \quad \sum_{j=1}^{J} \| \tilde{M}_j^{\rho,m} v \|_{l^2(\mathbb{Z}^d)} \leq \sqrt{\prod_{i=1}^{d} m_i} \, A(J) \ \| v \|_{l^2(\mathbb{Z}^d)} \ .$$

The proof of this lemma is very close to Lemma 5's. Using once again an Euclidean division, we have

$$\sum_{j=1}^{J} \sqrt{\sum_{n\in\mathbb{Z}^d} | \tilde{M}_j^{\rho,m} v(n) |^2}$$

$$= \sum_{j=1}^{J} \sqrt{\sum_{l_1=1}^{m_1} \cdots \sum_{l_d=1}^{m_d} \sum_{n\in\mathbb{Z}^d} \sup_{N\in\mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^{N} v(m(n+u_k)+l) - \frac{1}{N_j} \sum_{k=1}^{N_j} v(m(n+u_k)+l) \right|^2} \ .$$

This last expression is smaller than

$$\sum_{j=1}^{J} \sum_{l_1=1}^{m_1} \cdots \sum_{l_d=1}^{m_d} \sqrt{\sum_{n\in\mathbb{Z}^d} \sup_{N\in\mathcal{N}_\rho^j} \left| \frac{1}{N} \sum_{k=1}^{N} v(m(n+u_k)+l) - \frac{1}{N_j} \sum_{k=1}^{N_j} v(m(n+u_k)+l) \right|^2}$$

$$= \sum_{j=1}^{J} \sum_{l_1=1}^{m_1} \cdots \sum_{l_d=1}^{m_d} \sqrt{\tilde{M}_j^{\rho}(v(m.+l))(n)}.$$

But using the hypothesis of the lemma this is also smaller than

$$\begin{aligned} &\textstyle\sum_{l_1=1}^{m_1} \cdots \sum_{l_d=1}^{m_d} A(J) \ \| v(m.+l) \|_{l^2(\mathbb{Z}^d)} \\ &\quad = \textstyle\sum_{l_1=1}^{m_1} \cdots \sum_{l_d=1}^{m_d} A(J) \sqrt{\sum_{n\in\mathbb{Z}^d} | v(m\,n+l) |^2} \\ &\quad \leq A(J) \sqrt{\prod_{i=1}^{d} m_i} \sqrt{\sum_{l_1=1}^{m_1} \cdots \sum_{l_d=1}^{m_d} \sum_{n\in\mathbb{Z}^d} | v(m\,n+l) |^2} \\ &\quad = A(J) \sqrt{\prod_{i=1}^{d} m_i} \ \| v \|_{l^2(\mathbb{Z}^d)} \end{aligned}$$

◇

An immediate consequence of this is the next corollary

Corollary 5 *Let $(u_k)_{k\in\mathbb{N}}$ be a $\mathbb{Z}^d$-valued sequence such that there exists, for all fixed J, a number $A(J)$ independent of v and $(N_j)_{j\geq 1}$ which satisfies*

$$\forall v \in l^2(\mathbb{Z}^d), \quad \sum_{j=1}^{J} \| \tilde{M}_j^{\rho} v \|_{l^2(\mathbb{Z}^d)} \leq A(J) \ \| v \|_{l^2(\mathbb{Z}^d)} \ .$$

Let $\{\theta_N, \, N \in \mathbb{N}\}$ be a family of independent, identically distributed, $\mathbb{Z}^d$-valued random variables with $\mathbb{E} \mid \prod_{i=1}^{d} \theta_0^{(i)} \mid^{1/2} < \infty$. Let $M_j^{\rho} v$ be defined by (21) with $X_N = \theta_N \, u_N$. Then, we have

$$\forall v \in l^2(\mathbb{Z}^d), \quad \sum_{j=1}^{J} \| M_j^{\rho} v \|_{l^2(\mathbb{Z}^d)} \leq \left(\mathbb{E} \left| \prod_{i=1}^{d} \theta_0^{(i)} \right|^{1/2} \right) A(J) \ \| v \|_{l^2(\mathbb{Z}^d)} \ .$$

Let us denote once again $P_m = P(\{\omega \in \Omega : \theta_0(\omega) = m\})$. Then we have

$$\begin{array}{rcl} \tilde{M}_j^\rho v(n) & = & \sup_{N\in\mathcal{N}_\rho^j} \mid \frac{1}{N}\sum_{k=1}^N \sum_{m\in\mathbb{Z}^d} P_m v(n+m\,u_k) - \frac{1}{N_j}\sum_{k=1}^{N_j}\sum_{m\in\mathbb{Z}^d} P_m v(n+m\,u_k) \mid \\ & \leq & \sum_{m\in\mathbb{Z}^d} P_m \sup_{N\in\mathcal{N}_\rho^j} \mid \frac{1}{N}\sum_{k=1}^N v(n+m\,u_k) - \frac{1}{N_j}\sum_{k=1}^{N_j} v(n+m\,u_k) \mid \\ & = & \sum_{m\in\mathbb{Z}^d} P_m\, M_j^{\rho,m} v(n) \end{array}$$

and, using lemma 6,

$$\sum_{j=1}^J \parallel \tilde{M}_j^\rho v \parallel_{l^2(\mathbb{Z}^d)} \leq \sum_{m\in\mathbb{Z}^d} P_m \sqrt{\prod_{i=1}^d m_i\, A(J)} \parallel v \parallel_{l^2(\mathbb{Z}^d)} = (\mathbb{E} \mid \prod_{i=1}^d \theta_0^{(i)} \mid^{1/2})\, A(J) \parallel v \parallel_{l^2(\mathbb{Z}^d)} .$$

◇

Finally, we have found easy conditions on the process $\{X_N = \theta_N.u_N, N \in \mathbb{Z}\}$ to satisfy the hypotheses of Theorem 7. We just need the extra condition

$$\forall 0 \leq i \leq d, \qquad \mathbb{E}_\theta \mid \theta_N^{(i)} u_N^{(i)} \mid^\delta = O(2^{N^\gamma}),$$

which is satisfied as soon as

$$\forall 0 \leq i \leq d, \qquad \mathbb{E}_\theta \mid \theta_0^{(i)} \mid^\delta < \infty \text{ and } u_N = O(2^{N^{\gamma'}}),$$

for some $\gamma' < \gamma$. So we obtain the following theorem of convergence.

Theorem 8 *Let $\{\theta_n, n \in \mathbb{N}\}$ be a family of independent, identically distributed, $\mathbb{Z}^d$ valued random variables such that $\mathbb{E} \mid \prod_{i=1}^d \theta_0^{(i)} \mid^{1/2} < \infty$, and let $(u_n)_{n\in\mathbb{N}}$ be a MVI-good sequence on $\mathbb{Z}^d$ such that $\parallel u_n \parallel = O(2^{n^\gamma})$ (with $\gamma < 1$). Then there exists a absolute measurable set Ω_0 such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, for all measurable dynamical system $(Y, \mathcal{A}, \mu, T)$, T being a $\mathbb{Z}^d$-action,*

$$\forall f \in L^2(\mu), \qquad \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N f \circ T^{\theta_k(\omega)\, u_k} \text{ exists } \mu\text{-almost everywhere.}$$

Using the same arguments, we can find a result of [15] again: the convergence of the ergodic mean associated with $X_N = u_N + \theta_N$, where $\mathbb{E} \mid \prod_{i=1}^d \theta_0^{(i)} \mid^\delta < \infty$ for some $\delta > 0$. More generally, we have, the same way, the convergence of the ergodic mean associated with $X_N = \theta_N\, u_N + \tilde{\theta}_N$, where $(\theta_N)_{N\in\mathbb{N}}$ and $(\tilde{\theta_N})_{N\in\mathbb{N}}$ are two i.i.d. random processes (see Theorem 2).

Let us point out (in the one dimensionnal case $d = 1$) that the conditions on $(u_N)_{N\in\mathbb{N}}$ above, are satisfied by the sequences given in the introduction as, in particular, $u_N = [P(N)]$ the integer part of a $l-$degree polynomial $P(N)$ with real coefficients. This example raises the more general question of the convergence of ergodic means associated with the processes

$$X_N(\omega) = \sum_{k=1}^l \theta_N^{(k)}(\omega)\, N^k,$$

where $\{\theta_N^{(k)}, N \in \mathbb{N}\}$'s are sequences of independent, identically distributed, random variables. This open question is directly related to the problem of finding a bound for

$$\parallel \sup_{N\in\mathcal{N}} \mid (1/N) \sum_{k=1}^N v(. + P_l(k)) \mid\parallel_{l^2(\mathbb{Z}^d)} / \parallel v \parallel_{l^2(\mathbb{Z}^d)}$$

which depends only on the degree of the polynomial P_l.

4 Second application: study of the oscillation function

4.1 Proof of Theorem 3

We start with properties of oscillation functions associated with some probability system and non-negative contractions. Consider therefore a contraction T on $L^2(\mu)$, a function f in $L^2(\mu)$ and $\{u_k,\, k \geq 1\}$ a sequence of positive integers.

Proposition 7 *In the case when T is a non-negative contraction over $L^2(\mu)$ and $u_k = k$, we have, for all $f \in L^2(\mu)$,*

$$O_{L^2(\mu)}((T^k \circ f), (N_j), \mathcal{N}_\rho) \leq K||f||_{L^2(\mu)},$$

where K is an absolute constant.

Proof. This proposition arises from an estimation by Lifshits-Weber obtained by a spectral regularization principle ([19], Corollary 6.4.3, p. 110) in the case when T is a non-negative isometry on an arbitrary L^2-space, namely,

Theorem 9 *[19] Let T a non-negative isometry on $L^2(\mu)$, then, for all increasing sequence (N_j) of positive integers, there exists an absolute constant $K < \infty$ such that*

$$\sum_{j=1}^{\infty} \left|\left| \sup_{N_j < N \leq N_{j+1}} |A_N(f) - A_{N_j}(f)| \right|\right|^2_{L^2(\mu)} \leq K||f||^2_{L^2(\mu)}, \tag{24}$$

where $A_N(f) = (1/N)\sum_{k=1}^{N} T^k(f)$.

Moreover, the equation (24) remains true when the means A_N are generated by an arbitrary contraction on $L^2(\mu)$, under the extra condition $\sup_{j\geq 1} N_{j+1}/N_j < \infty$. *The constant K in (24) depends therefore on this quantity.*

The main difficulty consists, of course, in proving the preceding theorem. With the help of this theorem, the dilation theorem, where the argument of non-negativity is crucial, permits to prove Proposition 7.

Indeed, let $\mathcal{H} = L^2(\mu)$. By hypothesis, T is a non-negative contraction. In virtue of the dilation theorem, there exists another L^2-space that we denote $\hat{\mathcal{H}}$, and a non-negative isometry $\hat{T}$ such that, $DT^n = P\hat{T}^n D$, where $D : \mathcal{H} \to \hat{\mathcal{H}}$ is a non-negative isometric embedding from $\hat{\mathcal{H}}$ to $\mathcal{H}$ and $P : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ a non-negative projection. The non-negativity of the mentioned operators and the fact that a non-negative isometry on a L^2-space transforms functions with disjointed supports into functions with disjointed supports (see [10], p.186) leads to

$$O_{\mathcal{H}}(T, f, (N_j), \mathcal{N}_\rho) \leq 2O_{\hat{\mathcal{H}}}(\hat{T}, Df, (N_j), \mathcal{N}_\rho).$$

Then, finally, Weber's result states that

$$\forall f \in \mathcal{H}, \quad O_{\hat{\mathcal{H}}}(\hat{T}, Df, (N_j), \mathcal{N}_\rho) \leq K||Df||_{\hat{\mathcal{H}}} = K||f||_{\mathcal{H}}.$$

Hence the conclusion. ◇

Remarks:

1. Proposition 7 gives us directly the theorem of ergodic domination by Akcoglu (Maximal inequality) of an ergodic mean over a L^2-space for non-negative contractions ([10], Th.2.5, p.189).

2. Proposition 5 gives us directly Akcoglu's ergodic theorem for non-negative contractions defined on a L^2-space ([10]. Th.2.6, p.190).

We shall use Proposition 7 in order to bound the oscillation function of ergodic means along some random sequences.

Proposition 8 *Let $(\Omega, \mathcal{B}, P)$ be a complete probability space on which is defined a sequence of integer-valued independent random variables $\{X_k,\, k \in \mathbb{N}\}$. Moreover, assume that the law of the X_k's is generated by the convolution of a given law, that is to say, there exists an integrable random variable, say Y, such that*

$$\forall k \geq 1, \quad d\mathbb{P}_{X_k} = d\mathbb{P}_Y^{*(S_k)},$$

where (S_k) is a random walk generated by a sequence of non-centered i.i.d. random variables which have a moment of order two, and are defined on another space $(\Omega', \mathcal{B}', P')$ independent of $(\Omega, \mathcal{B}, P)$.

Then there exists a measurable set $\Omega_0 \subset \Omega \times \Omega'$, with $P \times P'(\Omega_0) = 1$, such that, for all fixed $(\omega, \omega') \in \Omega_0$, we have:

For all measurable dynamical system $(X, \mathcal{A}, \mu, T)$, and for all $f \in L^2(\mu)$, the oscillation function $O_{L^2(\mu)}(T, (f), (N_j), \mathcal{N}_\rho)$ associated with the sequence of ergodic means

$$\left\{ \frac{1}{N} \sum_{k=1}^{N} f \circ T^{X_k},\, N \geq 1 \right\},$$

satisfies

$$O_{L^2(\mu)}(T, (f), (N_j), \mathcal{N}_\rho) \leq K(\omega, \omega')\, ||f||_{L^2(\mu)},$$

where $K(\omega, \omega') < \infty$, $P \times P'$-a.e.

To prove this result, one of the tools we shall use consists in the following proposition that comes from a collective work of Lacey, Petersen, Rudolph and Wierdl [13].

Proposition 9 *Given a 1-dimensional, non-centered random walk which has a moment of order two, we know that there exist a set Ω_0 dense in Ω and an $\epsilon > 0$ such that, for all ω in that set, there exists $C_\omega < \infty$ such that,*

$$\forall N \geq 2, \quad \sup_{\alpha \in [-1/2, 1/2]} \left| \frac{1}{N} \sum_{k=1}^{N} \exp 2i\pi\alpha S_k(\omega) - \mathbb{E} \frac{1}{N} \sum_{k=1}^{N} \exp 2i\pi\alpha S_k \right| \leq \frac{C_\omega}{N^\epsilon}.$$

Let us prove Theorem 3.

Proof. First of all, recall that, for $\rho > 1$ and a given increasing sequence of positive integers (N_j), we denote

$$\mathcal{N}_\rho^j := \{[\rho^k],\, k \in \mathbb{N},\, N_j \leq [\rho^k] < N_{j+1}\}.$$

Denote also

$$A_N = \frac{1}{N} \sum_{k=1}^{N} T^{X_k}(f).$$

We suggest a proof in three steps.

-Step 1: Fix $\omega' \in \Omega'$. Hence, we assume that a realization of the law of the process (X_k) is given with the help of the sequence of positive number $\{S_k(\omega'),\, N \geq 1\}$. ¿From the application of the strong law of large numbers to the sequence (S_k) (that is to say, $\sup_{k \geq 1} S_k/k < \infty$, P'-a.e.), we can omit to suppose that, for all ω',

$$\mathbb{E}_\omega |X_k| = O(k).$$

The arguments used here are, on one hand, the hypothesis of integrability of the law Y and, on the other hand, the independence between the spaces on which are defined Y and the random walk (S_k).

We need to introduce the following quantity: $\mathbb{E}_\omega A_N$. It gives us the following immediate bound,

$$\left|\left| \sup_{N \in \mathcal{N}_\rho^j} |A_N - A_{N_j}| \right|\right|_{L^2(\mu)}^2 \leq 3 \sum_{N \in \mathcal{N}_\rho^j} ||A_N - \mathbb{E}_\omega A_N||_{L^2(\mu)}^2 + 3||A_{N_j} - \mathbb{E}_\omega A_{N_j}||_{L^2(\mu)}^2$$

$$+3 \left|\left| \sup_{N \in \mathcal{N}_\rho^j} |\mathbb{E}_\omega A_N - \mathbb{E}_\omega A_{N_j}| \right|\right|_{L^2(\mu)}^2 .$$

¿From Corollary 1 and Theorem 1, we have therefore

$$\sum_{N\in\mathcal{N}_\rho^j} ||A_N - \mathbb{E}_\omega A_N||^2_{L^2(\mu)} \le K(\omega,\omega')^2 \sum_{N\in\mathcal{N}_\rho^j} \frac{\log N}{N},$$

and

$$||A_{N_j} - \mathbb{E}_\omega A_{N_j}||^2_{L^2(\mu)} \le K(\omega,\omega')^2 \frac{\log N_j}{N_j},$$

where $K(\omega,\omega') < \infty$ for almost every ω and ω'.

Then, using the arguments detailed in the prove of Proposition 10, we show that

$$\left[\sum_{N\in\mathcal{N}_\rho^j} \frac{\log n_N}{N} + \frac{\log n_{N_j}}{N_j}\right]$$

is the general term of a convergent series of index j, and this happen independently of the lacunary sequence (N_j), for all geometric partial index $\mathcal{N}_\rho$.

It gives us a bound for the oscillation function we are interested in,

$$O_{L^2(\mu)}((T^{X_k}), f, (N_j), \mathcal{N}_\rho) \le ||f||_{L^2(\mu)} K(\omega,\omega') + 3\, O_{L^2(\mu)}((\mathbb{E}_\omega T^{X_k}), (f), (N_j), \mathcal{N}_\rho).$$

-Step 2: The “transfered” problem consists therefore in finding an estimation for

$$O_{L^2(\mu)}((\mathbb{E}_\omega T^{X_k}), f, (N_j), \mathcal{N}_\rho).$$

Denote $\tilde{T}$ the non-negative contraction on $L^2(\mu)$ defined by $\tilde{T}(.) = \mathbb{E}\, T^Y(.)$. Then the sequence of non-negative operators on $L^2(\mu)$ defined by $\mathbb{E}\, T^{X_k}(.)$ can be rewritten, without difficulty, in

$$\forall k \ge 1, \quad \mathbb{E}_\omega T^{X_k}(.) = \tilde{T}^{S_k(\omega')}(.).$$

Hence, in order to bound $O_{L^2(\mu)}((\tilde{T}^{S_k(\omega')}), f, (N_j), \mathcal{N}_\rho)$, we need to introduce the quantity $\mathbb{E}_{\omega'} \tilde{T}^{S_k}$. The technique used in the first step of the proof, together with, this time, Proposition 9 and the spectral lemma applied to the contraction $\tilde{T}$, gives us

$$O_{L^2(\mu)}((\tilde{T}), (f), (N_j), \mathcal{N}_\rho) \le ||f||_{L^2(\mu)} K(\omega,\omega') + 3\, O_{L^2(\mu)}((\mathbb{E}_{\omega'} \tilde{T}^{S_k(\omega')}), (f), (N_j), \mathcal{N}_\rho),$$

where $K(\omega,\omega') < \infty$ for almost every ω and ω'.

-Step 3: The problem “transfered” a second time consists therefore in finding an estimation for

$$O_{L^2(\mu)}((\mathbb{E}_{\omega'} \tilde{T}^{S_k(\omega')}), f, (N_j), \mathcal{N}_\rho).$$

Denote $\tilde{\tilde{T}}$ the non-negative contraction on $L^2(\mu)$ defined by $\tilde{\tilde{T}}(.) = \mathbb{E}\, \tilde{T}^{S_1}(.)$ where S_1 denotes the i.i.d increment of the random walk.

Hence, the sequence of contractions $\mathbb{E}_{\omega'} \tilde{T}^{S_k}(.)$ on $L^2(\mu)$ can be written as

$$\forall k \ge 1, \quad \tilde{\tilde{T}}^k(.) = \mathbb{E}_{\omega'} \tilde{T}^{S_k}(.).$$

Using Proposition 7, we show that

$$O_{L^2(\mu)}((\mathbb{E}_{\omega'} \tilde{T}^{S_k(\omega')}), (f), (N_j), \mathcal{N}) \le K ||f||_{L^2(\mu)},$$

where K is an absolute constant. This concludes the proof of Proposition 8. $\diamond$

This means, in particular, that, according to Proposition 5, $P \times P'$-almost everywhere, the sequence (X_k) is L^2-good.

We have, in fact, the expression of the limit which is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f \circ T^{X_k(\omega)} = \mathbb{E}(f|\mathcal{F}),\ \mu\text{-a.e.}$$

where $\mathcal{F}$ is the σ-field generated by the functions in $L^2(\mu)$ which are invariant under the action of T. To show this result, we just need to observe that the function which are invariant under T are the same as the ones under $\tilde{\tilde{T}}$. We conclude using Hopf’s ergodic theorem.

4.2 Sketch of the proof of Theorem 4

We start as in the first step of the proof of Theorem 3. Then we show that one can extend, without difficulty, the maximal and variational inequalities for non-negative contractions. We leave the details to the reader.

Remark 1: In the case when, for instance, u_k is a polynomial or the sequence of ordered primes numbers, we know that the variational inequalities are satisfied in the framework of non-negative isometries (see [2]).

Remark 2: In the case when $u_k = k$ and $\mathcal{L}(Y)$ is a Poisson law of parameter 1, a direct calculation gives an estimation of the oscillation function of the ergodic mean above, by the application of Weber's result, without any argument of dilation.

Indeed, for all $N \in \mathbb{N}$, let X_N be a Poisson of parameter N. We can easily check that such a process $\{X_N : N \in \mathbb{N}\}$ satisfies the condition (3) with $q_N = \sqrt{N \log N}$. Corollary 1 together with a reasoning analogous to the one made in the proof of Corollary 3, show therefore that, for all $\omega \in \Omega_0$, $(Y, \mathcal{A}, \mu, T)$ and $f \in L^2(\mu)$, the a.e. convergence of $(1/N)\sum_{k=1}^N f \circ T^{X_k}$ is equivalent to the one of $(1/N)\sum_{k=1}^N \mathbb{E} f \circ T^{X_k}$.

We shall focus now on the convergence of that last mean. The method is nearly the same. Since the ergodic mean $(1/N)\sum_{k=1}^N f \circ T^k$ is known to converge a.e. for all $f \in L^2(\mu)$, we just need to show that there exists a sequence a_N such that, for all $\rho > 1$, $\sum_{n \in \mathcal{N}_\rho} a_N < \infty$ and

$$\left\| \frac{1}{N}\sum_{k=1}^N \mathbb{E} f \circ T^{X_k} - \frac{1}{N}\sum_{k=1}^N f \circ T^k \right\|_{L^2(\mu)} \leq C \parallel f \parallel_{L^2(\mu)} a_N.$$

¿From the Spectral Lemma, we know that the left side of this last inequality is equivalent to

$$\left(\int_{[-1/2,1/2)} \left| \frac{1}{N}\sum_{k=1}^N \mathbb{E}\, e^{2\pi i \alpha X_k} - \frac{1}{N}\sum_{k=1}^N e^{2\pi i \alpha k} \right|^2 \mu_f(d\alpha) \right)^{1/2} \leq I_1 + I_2 + I_3,$$

where μ_f still denotes the spectral measure of the operator at the point f, and

$$I_1 := \left(\int_{|\alpha|<\epsilon} \left| \frac{1}{N}\sum_{k=1}^N \mathbb{E}\, e^{2\pi i \alpha X_k} - \frac{1}{N}\sum_{k=1}^N e^{2\pi i \alpha k} \right|^2 \mu_f(d\alpha) \right)^{1/2},$$

$$I_2 := \left(\int_{|\alpha| \in (\epsilon,1/2)} \left| \frac{1}{N}\sum_{k=1}^N \mathbb{E}\, e^{2\pi i \alpha X_k} \right|^2 \mu_f(d\alpha) \right)^{1/2},$$

$$I_3 := \left(\int_{|\alpha| \in (\epsilon,1/2)} \left| \frac{1}{N}\sum_{k=1}^N e^{2\pi i \alpha k} \right|^2 \mu_f(d\alpha) \right)^{1/2},$$

for some arbitrary $\epsilon \in]0, 1/2[$. To bound I_1 from above, we use the well-known property of Poisson laws,

$$\mathbb{E}\, e^{2\pi i \alpha X_k} = e^{k(e^{2\pi i \alpha}-1)}.$$

We have therefore to bound

$$\begin{aligned} \left| \tfrac{1}{N}\textstyle\sum_{k=1}^N \left(e^{k(e^{2\pi i\alpha}-1)} - e^{2\pi i \alpha k} \right) \right| &\leq \tfrac{1}{N}\textstyle\sum_{k=1}^N \left| e^{k(e^{2\pi i\alpha}-1)} - e^{2\pi i \alpha k} \right| \\ &\leq \tfrac{1}{N}\textstyle\sum_{k=1}^N \left| e^{k\,O(\alpha^2)} - 1 \right| \\ &\leq \tfrac{1}{N}\textstyle\sum_{k=1}^N k\, O(\alpha^2) \\ &\leq N\, O(\alpha^2) \\ &\leq N\, O(\epsilon^2). \end{aligned}$$

The last inequality comes from the fact that the integration is over $[-\epsilon, \epsilon]$. We obtain finally

$$I_1 \leq C \parallel f \parallel_{L^2(\mu)} N\epsilon^2.$$

To bound I_2 and I_3 from above, we use, respectively

$$\left| \frac{1}{N} \sum_{k=1}^{N} e^{k(e^{2\pi i\alpha}-1)} \right| \leq \frac{C}{N \mid \alpha \mid},$$

and

$$\left| \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i\alpha k} \right| \leq \frac{C}{N \mid \alpha \mid}.$$

We obtain easily

$$I_2 + I_3 \leq \frac{C}{N\epsilon} \parallel f \parallel_{L^2(\mu)},$$

since the integration is done, this time, over $\{\alpha \in [-1/2, 1/2] : \mid \alpha \mid > \epsilon\}$. Finally, putting $\epsilon = N^{-2/3}$, we have the desired bound,

$$\left(\int_{[-1/2,1/2)} \left| \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}\, e^{i2\pi\alpha X_k} - \frac{1}{N} \sum_{k=1}^{N} e^{i2\pi\alpha k} \right|^2 \mu_f(d\alpha) \right)^{1/2} \leq C \parallel f \parallel_{L^2(\mu)} N^{-1/3}.$$

Observe that we have also proved that the limit of $(1/N) \sum_{k=1}^{N} f \circ T^{X_k}$ is the same as $(1/N) \sum_{k=1}^{N} f \circ T^k$ which is known to be $\mathbb{E}(f \mid \mathcal{F})$ where $\mathcal{F}$ denotes the σ-field of invariants under T .

5 Third application: weighted ergodic means (proof of Theorem 5)

We end this work by a study of the following problem: given a probability space $(\Omega, \mathcal{B}, P)$ on which is defined a sequence of centered, independent and identically distributed random variable $(X_k,\, k \geq 1)$ that satisfies a condition on the moments (see below), given an increasing sequence of integers $(u_k,\, k \geq 1)$, given a measurable dynamical system $(X, \mathcal{A}, \mu, T)$, for $f \in L^2(\mu)$, what can we say about the pointwise convergence of the sequence of modulated ergodic means

$$\left\{ \frac{1}{N} \sum_{k=1}^{N} X_k f \circ T^{u_k},\, N \geq 1 \right\}?$$

Let us specify that

1. we do not assume that the sequence of random weight is bounded,
2. the sequence $(u_k,\, k \geq 1)$ is not supposed to be L^2-good for the pointwise ergodic theorem. Thus, we shall be able, in no way, to use any maximal inequality which is known to be true, for instance, when the sequence $(u_k,\, k \geq 1)$ is L^2-good.

In a previous work with Weber[17], one of the authors shows that when the sequence $(u_k,\, k \geq 1)$ is L^2-good and when the sequence of weights is non-negative, for all $f \in L^2(\mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} X_k f \circ T^{u_k} \text{ exists } \mu\text{-almost everywhere},$$

universally on the space where the sequence (X_k) is defined.

More recently, Assani[1] proved a result of the same kind in the case when $u_k = k$, the weights are centered and the spaces are of L^p-style, $p > 1$.

Given a sequence of centered weights, say (X_k), writing it as a difference of two non-negative sequences, $X_k = X_k.I_{X_k \geq 0} - (-X_k.I_{X_k<0})$, and applying [17] to both non-negative weights, we obtain immediately Assani's result for $p = 2$. It is therefore not difficult to observe that we have also the result for all $p > 1$.

To prove Theorem 5, we start by determining the oscillation function associated with the sequence of operators, $T_k := X_k T^{u_k}$. For that, recall that

$$O_{L^2(\mu)}((T_k), f, (N_j), \mathcal{N}) = \sqrt{\sum_{j\geq 1} \left|\left| \sup_{N\in\mathcal{N}_j} \left| \frac{1}{N}\sum_{k=1}^{N} T_k(f) - \frac{1}{N_j}\sum_{k=1}^{N_j} T_k(f) \right| \right|\right|_{L^2(\mu)}^2},$$

where $\mathcal{N}$ is a partial index, (N_j) is a sequence of integers such that $N_{j+1} \geq 2N_j$, and $f \in L^2(\mu)$.

Proposition 10 *Under the hypotheses of the preceding theorem, there exists a $\mathcal{B}$-measurable set Ω_0, with $P(\Omega_0) = 1$, such that, for all $\omega \in \Omega_0$, there exists a positive constant $K(\omega) < \infty$ such that, for all measurable dynamical system $(X, \mathcal{A}, \mu, T)$, and for all $f \in L^2(\mu)$,*

$$O_{L^2(\mu)}((T_k), (f), (N_j), \mathcal{N}) \leq K(\omega) ||f||_{L^2(\mu)}.$$

Moreover, the constant K is independent of the sequence (N_j), and we can chose $\mathcal{N}$ among the following indexes: $\mathcal{N}_\rho$ for $\rho > 1$ (geometrical index, see Section 1), $\mathcal{N}^\gamma = \{k^\gamma, k \geq 1\}$ where $\gamma > 1/(1-\beta)$ is an integer (β is given in the theorem).

A consequence of Proposition 10, is that we have the theorem with only partial indexes, and, a priori, we cannot obtain immediately the total index by arguments of non-negativity and boundedness (see Proposition 5). Actually, the lemma below permits to conclude. Indeed, it exhibits a connection between the sequence of random variables (X_k) and the index $\mathcal{N}^\gamma = \{k^\gamma, k \geq 1\}$ where $\gamma \geq 3$ is an integer.

Lemma 7 *Under the hypotheses of Proposition 10, for all integer $\gamma \geq 3$, we have*

$$\mathbb{E} \sup_{N\geq 1} \frac{1}{N^{\gamma-1}} \sum_{k=N^\gamma}^{(N+1)^\gamma} |X_k| < \infty.$$

Remark: The sequence does not need to be centered.

Let us prove, first, Proposition 10.

Proof. We suggest a proof relying on Proposition 6 and the spectral lemma. Let us first introduce a notation: for all integer N and all $f \in L^2(\mu)$, denote

$$A_N := \frac{1}{N}\sum_{k=1}^{N} T_k(f).$$

Hence, we have

$$\left|\left| \sup_{N\in\mathcal{N}^j} |A_N - A_{N_j}| \right|\right|_{L^2(\mu)}^2 \leq 2 \sum_{N\in\mathcal{N}^j} ||A_N||_{L^2(\mu)}^2 + 2||A_{N_j}||_{L^2(\mu)}^2,$$

where $\mathcal{N}^j := \{N \in \mathcal{N}, N_j \leq N < N_{j+1}\}$. Then, using the spectral lemma and Proposition 6, we obtain

$$\left|\left| \sup_{N\in\mathcal{N}^j} |A_N - A_{N_j}| \right|\right|_{L^2(\mu)}^2 \leq 2\, C(\omega)\, ||f||_{L^2(\mu)}^2 \left[\sum_{N\in\mathcal{N}^j} \frac{\log u_N}{N} + \frac{\log u_{N_j}}{N_j} \right].$$

¿From the hypothesis of increase made on (u_k), and the fact that $N_j \geq 2^j$, we have the estimation

$$\left|\left| \sup_{N\in\mathcal{N}^j} |A_N - A_{N_j}| \right|\right|_{L^2(\mu)}^2 \leq 2\, C(\omega)\, ||f||_{L^2(\mu)}^2 \left[\tilde{C} \sum_{N\in\mathcal{N}^j} \frac{1}{N^{1-\beta}} + \frac{1}{2^{j(1-\beta)}} \right],$$

where $\tilde{C}$ is a constant which depends only on the sequence (u_k).

- In the case when $\mathcal{N} = \mathcal{N}_\rho = \{[\rho^k],\, k \geq 1\}$ with $\rho > 1$, it is clear that we obtain

$$O_{L^2(\mu)}((T_k), f, (N_j), \mathcal{N}_\rho) \leq K(\rho, (u_k), \omega)||f||_{L^2(\mu)},$$

with $K(\rho, (u_k), \omega) < \infty$ for all $\rho > 1$.

- In the case when $\mathcal{N} = \mathcal{N}^\gamma = \{k^\gamma,\, k \geq 1\}$ with $\gamma > 1/(1-\beta)$ an integer, it is clear that we obtain

$$O_{L^2(\mu)}((T_k), f, (N_j), \mathcal{N}_\gamma) \leq K(\gamma, (u_k), \omega)||f||_{L^2(\mu)},$$

with $K(\gamma, (u_k), \omega) < \infty$ for all $\gamma > 1/(1-\beta)$. Hence the proof of Proposition 10.

$\diamond$

Next, we give a proof of the lemma which relies on an argument of Gaussian randomization.
Proof. Introducing the quantity $\mathbb{E}|X_k|$, we have the following bound

$$\mathbb{E} \sup_{N\geq 1} \frac{1}{N^{\gamma-1}} \sum_{k=N^\gamma}^{(N+1)^\gamma} |X_k| \leq \mathbb{E} \sup_{N\geq 1} \frac{1}{N^{\gamma-1}} \left| \sum_{N^\gamma}^{(N+1)^\gamma} |X_k| - \mathbb{E}|X_k| \right| + C(\gamma)\mathbb{E}|X_1|,$$

where $C(\gamma)$ is a constant which depends on γ.

¿From symetrization and contraction principles, it remains us to prove that

$$\mathbb{E}_{X,g} \sup_{N\geq 1} \frac{1}{N^{\gamma-1}} \left| \sum_{k=N^\gamma}^{(N+1)^\gamma} g_k|X_k| \right| < \infty,$$

where $\{g_k,\, k \geq 1\}$ is an isonormal sequence.

Fix the variable X, and integrate over g, the Gaussian random variable indexed on positive integers, $G_N := \sum_{k=N^\gamma}^{(N+1)^\gamma} g_k|X_k|$. For all fixed X and all N, denote

$$\Gamma_N = \frac{1}{\sqrt{\sum_{k=N^\gamma}^{(N+1)^\gamma} |X_k|^2}} G_N.$$

Note that, for all N, $\mathcal{L}(\Gamma_N) = \mathcal{N}(0,1)$. Therefore,

$$\mathbb{E}_{X,g} \sup_{N\geq 1} \frac{1}{N^{\gamma-1}} |G_N| \leq \mathbb{E}_X 2^\gamma \sup_{M\geq 1} \sqrt{\frac{1}{M}\sum_{k=1}^{M} |X_k|^2} \left[\mathbb{E}_g \sup_{N\geq 1} \left| \frac{1}{N^{\gamma/2-1}} \Gamma_N \right| \right].$$

As $\gamma/2 - 1 > 0$ ($\gamma \geq 3$ by hypothesis), we know that $\mathbb{E}_g \sup_{\geq 1} |\Gamma_N/N^{\gamma/2-1}| < \infty$. And the condition on the moments imposed to the sequence (X_k) leads to the conclusion that

$$\mathbb{E}_X 2^\gamma \sup_{M\geq 1} \sqrt{\frac{1}{M}\sum_{k=1}^{M} |X_k|^2} < \infty.$$

$\diamond$

Finally, we prove Theorem 5
Proof. ¿From Proposition 10, we know that, for all dynamical system and all $f \in L^2(\mu)$, on the indexes $\mathcal{N}_\gamma = \{k^\gamma,\, k \geq 1\}$ with γ an integer large enough (which depends on the increase of the sequence (u_k)), we have

$$\lim_{N\to\infty} \frac{1}{N^\gamma} \sum_{k=1}^{N^\gamma} X_k(\omega) f(T^{u_k}x) = 0, \ \mu\text{-a.e.}$$

The lemma permits us to show the existence of a positive, random variable C which is integrable and such that, for all $N \geq 1$, we have

$$\frac{1}{N^\gamma} \sum_{k=N^\gamma}^{(N+1)^\gamma} |X_k(\omega)| \leq \frac{C(\omega)}{N}.$$

For all positive integer M, there exists an integer N such that

$$N^\gamma \leq M < (N+1)^\gamma.$$

We have the following inequality

$$\left| \frac{1}{M} \sum_{k=1}^{M} X_k(\omega) f(T^{u_k} x) \right| \leq$$

$$\frac{N^\gamma}{M} \left| \frac{1}{N^\gamma} \sum_{k=1}^{N^\gamma} X_k(\omega) f(T^{u_k} x) \right| + \frac{1}{N^\gamma} \sum_{k=N^\gamma}^{(N+1)^\gamma} |X_k(\omega)| \, |f(T^{u_k} x)|.$$

The first term converges to 0 when N tends to ∞. Let us focus on the second term. Without loss of generality, we can assume that f is non-negative. Then, we decompose f as follows, $f = f_1 + f_2 = f\,\mathbb{I}_{f \leq \sqrt{N}} + f\,\mathbb{I}_{f > \sqrt{N}}$.

For the first term, we have,

$$\frac{1}{N^\gamma} \sum_{k=N^\gamma}^{(N+1)^\gamma} |X_k(\omega)| \, |f_1(T^{u_k} x)| \leq C(\omega) \, \frac{\sqrt{N}}{N}$$

which tends to 0 when N tends to ∞, because of the lemma.

For the second term, by integrating with the measure μ which is invariant under the action of the operator T, we have again, from the lemma,

$$\frac{1}{N^\gamma} \sum_{k=N^\gamma}^{(N+1)^\gamma} |X_k(\omega)| \int |f_2(T^{u_k} x)| \mu(dx) \leq C(\omega) \int \frac{f\,\mathbb{I}_{f > \sqrt{N}}}{N} \, d\mu. \tag{25}$$

We finish by remarking that

$$\sum_{N \geq 1} \int \frac{f\,\mathbb{I}_{f > \sqrt{N}}}{N} \, d\mu < \infty.$$

Indeed, applying Schwarz inequality to the sum and the integral, we obtain

$$\sum_{N \geq 1} \int \frac{f\,\mathbb{I}_{f < \sqrt{N}}}{N} \, d\mu \leq \sqrt{\int \sum_{N \geq 1} \frac{f^2}{N^2} \, d\mu} \sqrt{\sum_{N \geq 1} \mu(f^2 > N)} < \infty,$$

because f is in $L^2(\mu)$.

Remark finally that the same calculations would lead to the same kind of result in L^p. $\diamond$

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